Lecture One: Harmonic Functions and the Harnack Inequality

1 The Laplacian

Let Ω be an open subset of $\mathbb{R}^n$, and let $u : \Omega \to \mathbb{R}$ be a smooth function. We define the Laplacian by

$$\triangle u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}. \tag{1}$$

The equation

$$\triangle u = 0 \tag{2}$$

is called the Laplace equation, and functions which satisfy it are said to be harmonic. Harmonic functions turn out to be very important, and much of this course will be devoted to their study. Also of interest are functions with non-negative or non-positive laplacian. These are termed sub- and super-harmonic respectively.

2 The Maximum principle

The maximum principle is simply the statement that the gradient of a function at a maximum is zero. Formally, if u is a twice differentiable function on a closed ball $\bar{B}_r(x)$ with a maximum at x, then

$$\nabla u = 0 \text{ at x and } \triangle u \leq 0 \text{ at x.} \tag{3}$$

The one dimensional case should be familiar, and proofs of other cases are analogous.

3 Dirichlet Energy

Recall that the Dirichlet energy of a function $v : \Omega \to \mathbb{R}$ is given by $\int_\Omega |\nabla v|^2$. We will show that harmonic functions correspond to critical points of Dirichlet energy. For fixed v and for any $\phi \in C_0^\infty(\Omega)$ (i.e. ϕ an infinitely differentiable real valued function on Ω) we define

$$E_\phi(t) = \int_\Omega |\nabla(v + t\varphi)|^2. \tag{4}$$

Now compute

$$E_\phi(t) = \int_\Omega |\nabla v|^2 + t^2 \int_\Omega |\nabla \phi|^2 + 2t \int_\Omega \nabla v \cdot \nabla \phi,$$

so

$$\frac{d}{dt}|_{t=0} E_\phi(t) = 2 \int_\Omega \nabla v \cdot \nabla \phi. \tag{5}$$

Since $\phi = 0$ on $\partial\Omega$, it is clear that $\int_{\partial\Omega} \phi \nabla v \cdot dS = 0$. Applying Stokes' theorem to this gives

$$\int_\Omega \nabla v \cdot \nabla \phi = -\int_\Omega \phi \triangle v,$$

which we apply to (5) to get

$$\frac{d}{dt}|_{t=0} E_\phi(t) = -2 \int_\Omega \phi \triangle v. \tag{6}$$

Thus, if $\triangle v = 0$, then $\frac{d}{dt}|_{t=0} E_\phi(t) = 0$ for all ϕ. The converse is also true but we will not prove it here. These results give our correspondence. The following proposition makes it more explicit.

Proposition 3.1 *Let $\Omega \subset \mathbb{R}^n$ be open. If u is a harmonic function on Ω then*

$$\int_\Omega |\nabla u|^2 \leq \int_\Omega |\nabla v|^2, \tag{7}$$

for all functions v satisfying $u = v$ on $\partial\Omega$. In other words harmonic functions have the smallest Dirichlet energy for their boundary values.

Proof is by calculation. Clearly $\int_{\partial\Omega} (v - u) \nabla (v + u) \cdot dS = 0$. Therefore, by Stokes' theorem,

$$\int_\Omega \nabla (v - u) \cdot \nabla (v + u) = -\int_\Omega (v - u) \triangle (v + u). \tag{8}$$

Similarily

$$\int_\Omega \nabla (v - u) \cdot \nabla (v - u) = -\int_\Omega (v - u) \triangle (v - u). \tag{9}$$

Apply these to get

$$
\begin{aligned}
\int_\Omega |\nabla v|^2 - |\nabla u|^2 &= \int_\Omega \nabla(v-u)\cdot\nabla(v+u) \\
&= -\int_\Omega (v-u)\triangle(v+u) \\
&= -\int_\Omega (v-u)\triangle(v-u) \\
&= \int_\Omega |\nabla(v-u)|^2 \\
&\geq 0
\end{aligned}
$$

as required. ∎

4 The Mean Value Property

The following property of harmonic properties turns out to be very useful

Theorem 4.1 *Let $x \in \mathbb{R}^n$ and take $B_{r_0}(x)$ a ball around x. If u is a harmonic function on $B_{r_0}(x)$ then*

$$u(x) = \frac{1}{vol\ \partial B_r(x)} \int_{\partial B_r(x)} u, \tag{10}$$

and

$$u(x) = \frac{1}{vol\ B_r(x)} \int_{B_r(x)} u \tag{11}$$

for all $0 < r \leq r_0$.

Proof First note that

$$\frac{d}{dr}\frac{1}{\text{vol } \partial B_r(x)} \int_{\partial B_r(x)} u = \frac{1}{\text{vol } \partial B_r(x)} \int_{\partial B_r(x)} \frac{\partial u}{\partial \mathbf{n}}. \tag{12}$$

This is simply saying that the derivative of the average of u is the average of the outward normal derivative. Using this we can calculate

$$
\begin{aligned}
\frac{d}{dr}\frac{1}{\text{vol } \partial B_r(x)} \int_{\partial B_r(x)} u &= \frac{1}{\text{vol } \partial B_r(x)} \int_{\partial B_r(x)} \frac{\partial u}{\partial \mathbf{n}} \\
&= \frac{1}{\text{vol } \partial B_r(x)} \int_{\partial B_r(x)} \nabla u \cdot dS \\
&= \frac{1}{\text{vol } \partial B_r(x)} \int_{B_r(x)} \triangle u \\
&= 0
\end{aligned}
$$

by Stokes' theorem. Thus

$$\frac{1}{\text{vol } \partial B_r(x)} \int_{\partial B_r(x)} u$$

is constant on $0 < r \leq r_0$. This gives

$$u(x) = \lim_{s \to 0} \frac{1}{\text{vol } \partial B_s(x)} \int_{\partial B_s(x)} u = \frac{1}{\text{vol } \partial B_r(x)} \int_{\partial B_r(x)} u$$

as required. For the second statement calculate

$$\begin{aligned} \int_{B_r(x)} u &= \int_0^r \left(\int_{\partial B_s(x)} u \right) ds \\ &= \int_0^r u(x) \text{ vol } \partial B_s(x) ds \\ &= u(x) \text{ vol } B_r(x). \quad \blacksquare \end{aligned}$$

By a similar argument we can also show

$$u(x) \leq \frac{1}{\text{vol } \partial B_r(x)} \int_{\partial B_r(x)} u \text{ and } u(x) \leq \frac{1}{\text{vol } B_r(x)} \int_{B_r(x)} u \tag{13}$$

for sub-harmonic u, or

$$u(x) \geq \frac{1}{\text{vol } \partial B_r(x)} \int_{\partial B_r(x)} u \text{ and } u(x) \geq \frac{1}{\text{vol } B_r(x)} \int_{B_r(x)} u \tag{14}$$

for super-harmonic u. One consequence of the mean value property is the following.

Corollary 4.2 *If $x \in \mathbb{R}^n$ and u is a harmonic function on $B_r(x)$ then u takes both it's maximum and it's minimum value on the boundary $\partial B_r(x)$.*

Proof If u has no interior maximum then it's maximum must be on the boundary, and we're done. Else take y an interior maximum, and set $c = u(y)$. By the maximum principle c is the average of u over any sphere surrounding y. Since y is a maximum we also have $u \leq c$ on each of these spheres, and we conclude that $u = c$ on each sphere. Now take a sphere that intersects with the boundary, and u takes it's maximum on this intersection. The argument for the minimum is similar. $\blacksquare$

By a very similar argument we can extend this result to shapes other than spheres, show that subharmonic functions take their maximum on the boundary, and show that superharmonic functions take they're minimum on the boundary.

5 Harnack Inequality

Another useful property of harmonic functions is the Harnack inequality.

Theorem 5.1 *Let $B_{2r}(0)$ be an open ball in $\mathbb{R}^n$. There is a constant C depending only on the dimension n such that*

$$\sup_{B_r(0)} u \leq C \inf_{B_r(0)} u. \tag{15}$$

for all functions u that are non-negative and harmonic on $B_{2r}(0)$.

Proof Pick $x, y \in B_r(0)$. We must show that $u(x) \leq Cu(y)$. Let $d \leq 2r$ be the distance between x and y, and pick w and z one and two thirds of the way from x to y respectively. Note that u is positive and harmonic on $B_r(x)$ and that $B_{r/3}(w) \subset B_r(x)$. By the mean value property we have

$$\begin{aligned} u(w) &= \frac{1}{\text{vol } B_{r/3}(w)} \int_{B_{r/3}(w)} u \\ &= \frac{3^n}{\text{vol } B_r(x)} \int_{B_{r/3}(w)} u \\ &\leq \frac{3^n}{\text{vol } B_r(x)} \int_{B_r(x)} u \\ &\leq 3^n u(x). \end{aligned}$$

By a similar calculation we compare w, z and z, y to get

$$u(x) \leq 3^{3n} u(y) \tag{16}$$

as required. ∎

This is a very powerful result about harmonic functions, with several consequences. For example, if we have $u \geq 0$ and $\triangle u = 0$ on $B_{2r}(0)$ and $\inf_{B_r(0)} u = 0$, then u is identically 0 on $B_r(0)$. In fact, by modifying the above argument, we can obtain a similar result for any radius $s < 2r$.