An Improved Gradient Estimate for Harmonic Functions

1 The new gradient estimate

Last lecture we used an improved form of the gradient estimate for harmonic functions. We will now prove it.

Theorem 1.1 *There are dimensional constants c such that*

$$\sup_{B_r} \frac{|\nabla u|}{u} \leq \frac{c}{r} \tag{1}$$

for all positive harmonic functions $u : B_{2r} \to \mathbb{R}$.

Proof We will prove the result for $r = 1$ and claim that the general case follows immediately by scalng. As usual we take a non-negative test function $\phi : B_2 \to \mathbb{R}$ with $\phi = 0$ on ∂B_2. Define $v = \log u$ and $w = |\nabla v|^2$. Note that $\nabla v = \frac{\nabla u}{u}$ and $\triangle v = -\frac{|\nabla u|^2}{u^2} = -w$. We start by bounding $\triangle(w\phi^4) + 2\nabla v \cdot \nabla(w\phi^4)$ by a quartic polynomial in $w^{1/2}\phi$. Calculate

$$\begin{aligned} \triangle w &= \triangle|\nabla v|^2 \\ &= 2\left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)^2 + 2 < \nabla\triangle v, \nabla v > \\ &= 2\left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)^2 - 2 < \nabla w, \nabla v > \end{aligned}$$

by the Bochner formula. Therefore

$$\triangle(w\phi^4) = \phi^4\triangle w + 2\nabla\phi^4 \cdot \nabla w + w\triangle\phi^4 \tag{2}$$

$$= 2\phi^4\left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)^2 - 2\phi^4 < \nabla w, \nabla v > +2\nabla\phi^4 \cdot \nabla w + w\triangle\phi^4. \tag{3}$$

Now try to find our quartic bound. Consider

$$\triangle(w\phi^4) + 2\nabla v \cdot \nabla(w\phi^4) = \triangle(w\phi^4) + 2\phi^4\nabla v \cdot \nabla w + 2w\nabla v \cdot \nabla\phi^4. \tag{4}$$

Substitute for $\triangle(w\phi^4)$ from (3) to get

$$\triangle(w\phi^4) + 2\nabla v \cdot \nabla(w\phi^4) = 2\phi^4 \left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)^2 + 2\nabla\phi^4 \cdot \nabla w + w\triangle\phi^4 + 2w\nabla v \cdot \nabla\phi^4. \quad (5)$$

We need to write the second term out in terms of partial derivatives. Calculate

$$\begin{aligned} 2\nabla\phi^4 \cdot \nabla w &= 2\frac{\partial\phi^4}{\partial x_i}\frac{\partial|\nabla v|^2}{\partial x_i} \\ &= 4\frac{\partial\phi^4}{\partial x_i}\frac{\partial v}{\partial x_j}\frac{\partial^2 v}{\partial x_j \partial x_i}, \end{aligned}$$

so

$$\triangle(w\phi^4) + 2\nabla v \cdot \nabla(w\phi^4) = 2\phi^4 \left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)^2 + 4\frac{\partial\phi^4}{\partial x_i}\frac{\partial v}{\partial x_j}\frac{\partial^2 v}{\partial x_j \partial x_i} + w\triangle\phi^4 + 2w\nabla v \cdot \nabla\phi^4. \quad (6)$$

Use an absorbing inequality to simplify. Let $a_{ij} = \phi^2 \frac{\partial^2 v}{\partial x_i \partial x_j}$ and $b_{ij} = \frac{\partial\phi^4}{\partial x_i}\frac{\partial v}{\partial x_j}$. Note that $a_{ij}^2 + 4a_{ij}b_{ij} \geq -4b_{ij}^2$. Together with (6) we have

$$\begin{aligned} \triangle(w\phi^4) + 2\nabla v \cdot \nabla(w\phi^4) &\geq \phi^4 \left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)^2 - 4b_{ij}^2 + w\triangle\phi^4 + 2w\nabla v \cdot \nabla\phi^4 \\ &\geq \phi^4 \left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)^2 - 4|\nabla\phi^4|^2|\nabla v|^2 + w\triangle\phi^4 + 2w\nabla v \cdot \nabla\phi^4 \\ &\geq \phi^4 \left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)^2 - 16\phi^6|\nabla\phi|^2|\nabla v|^2 + (4\phi^3\triangle\phi + 12\phi^2|\nabla\phi|^2)w - 8\phi^3 w|\nabla v||\nabla\phi| \end{aligned}$$

since ϕ and w are both non-negative. Observe that $\phi, |\nabla\phi|$, and $\triangle\phi$ are bounded, so there are constants c_1, c_2, c_3 such that

$$\triangle(w\phi^4) + 2\nabla v \cdot \nabla(w\phi^4) \geq \phi^4 \left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)^2 - c_1\phi^2|\nabla v|^2 + c_2\phi^2 w - c_3\phi^3 w|\nabla v|. \quad (7)$$

Recall that for any collection of real numbers the average of the squares is greater than the square of the average. Thus for any matrix A

$$\frac{\sum A_{ij}^2}{n} \geq \frac{\sum A_{ii}^2}{n} \geq \left(\frac{\sum A_{ii}}{n}\right)^2.$$

Apply this above to give

$$\left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)^2 \geq \frac{1}{n}\left(\sum \frac{\partial^2 v}{\partial x_i^2}\right)^2 = \frac{(\triangle v)^2}{n},$$

and substitute this into (7) ;

$$\triangle(w\phi^4) + 2\nabla v \cdot \nabla(w\phi^4) \geq \phi^4 \ \frac{(\triangle v)^2}{n} - c_1\phi^2|\nabla v|^2 + c_2\phi^2 w - c_3\phi^3 w|\nabla v|. \tag{8}$$

Observe that $\triangle v = -|\nabla v|^2 = -w$, so we have

$$\triangle(w\phi^4) + 2\nabla v \cdot \nabla(w\phi^4) \geq \phi^4 \ \frac{w^2}{n} - c_1\phi^2 w + c_2\phi^2 w - c_3\phi^3 w^{3/2}. \tag{9}$$

This is the bound we were looking for.

Now apply this. Since ϕ is zero on the boundary of B_2, $\phi^4 w$ takes it's maximum in the interior. Let x be the maximum. At x, $\nabla(\phi^4 w) = 0$ and $0 \geq \triangle(\phi^4 w)$, so

$$0 \ \geq \ \frac{1}{n}(\phi w^{1/2})^4 + (c_2 - c_1)(\phi w^{1/2})^2 - c_3(\phi w^{1/2})^3. \tag{10}$$

This is a quartic polynomial in $\phi w^{1/2}$ with positive leading coefficient. Such polynomials are positive for large argument, so there is a constant k with $|\phi(x)(w(x))^{1/2}| \leq k$. Note that k depends only on the coefficients of the polynomial. The coefficients themselves depend only on dimension, so k also depends only on dimension. Choose $0 \leq \phi \leq 1$. Then

$$\sup_{B_2} \phi^4 w = (\phi(x))^4 w(x) \leq (\phi(x)(w(x))^{1/2})^2 \leq k^2. \tag{11}$$

Finally we choose ϕ to be identically one on B_1, so

$$\begin{aligned} \sup_{B_1} \frac{|\nabla u|^2}{u^2} &= \sup_{B_1} \phi^4 w & (12) \\ &\leq \sup_{B_2} \phi^4 w & (13) \\ &\leq k^2 & (14) \end{aligned}$$

and take square roots to give our result. ∎

The gradient estimate we proved earlier follows easily from this; this is a stronger result. As we saw last time the Harnack inequality is also a reasonably straightforward consequence. The only major annoyance is that we needed $u > 0$.