The Heat equation.$[t_1 > t_2]$

We've spent a lot of time concentrating on the laplace equation, but there are other important PDE's. One example is the heat equation, which we will study in this lecture. Consider a function $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ of both time and space. The heat equation is

$$\triangle u = \frac{\partial u}{\partial t}. \tag{1}$$

In this lecture we will prove a gradient estimate and a Harnack inequality for functions satisfying the heat equation on a torus $T^n = S^1 \times S^1 \times \cdots \times S^1$, since this turns out to be easier than doing the proof for a ball.

1 A gradient estimate for a torus

Theorem 1.1 *If u is positive and satisfies the heat equation on the cylinder $T^n \times \mathbb{R}$ then*

$$\frac{|\nabla u|^2}{u^2} - \frac{1}{u}\frac{\partial u}{\partial t} \leq \frac{n}{2t}. \tag{2}$$

Proof For this proof we will use the notation $g_t = \frac{\partial g}{\partial t}$. Define $f = \log u$, and calculate

$$\begin{aligned}
(\triangle - \frac{\partial}{\partial t})f &= \frac{\triangle u}{u} - \frac{|\nabla u|^2}{u^2} - \frac{1}{u}\frac{\partial u}{\partial t} \\
&= -\frac{|\nabla u|^2}{u^2} \\
&= -|\nabla f|^2.
\end{aligned}$$

Also define $F = t(|\nabla f|^2 - f_t)$. Note that we actually want to bound $\frac{F}{t}$. We need to estimate $(\triangle - \frac{\partial}{\partial t})F$. Observe that

$$\begin{aligned}
\triangle F &= t(\triangle|\nabla f|^2 - \triangle f_t) &(3)\\
&= 2t\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)^2 + 2t < \nabla\triangle f, \triangle f > -t\triangle f_t &(4)
\end{aligned}$$

by Bochner. Recall that, for any matrix A_{ij}, $\sum_{i,j} A_{ij}^2 \geq \frac{(\sum_i A_{ii})^2}{n}$ (We saw this in lecture 10, and it essentially because the average of the square is greater than the square of the average). Therefore

$$\triangle F \geq \frac{2t(\triangle f)^2}{n} + 2t < \nabla \triangle f, \triangle f > - t \triangle f_t. \tag{5}$$

We also have $\triangle f = -|\nabla f|^2 + f_t = -\frac{F}{t}$, so

$$\triangle F \geq \frac{2F^2}{nt} - 2 < \nabla F, \nabla f > - t \triangle f_t. \tag{6}$$

Now work on F_t. Clearly

$$F_t = |\nabla f|^2 - f_t + t(2\nabla f \cdot \nabla f_t) - t f_{tt}. \tag{7}$$

Note that $\triangle f + |\nabla f|^2 = f_t$, so

$$\begin{aligned} F_t &= |\nabla f|^2 - f_t + t(2\nabla f \cdot \nabla f_t) - t(\triangle f + |\nabla f|^2)_t & (8) \\ &= |\nabla f|^2 - f_t - t\triangle f_t. & (9) \end{aligned}$$

Putting together (6) and (9) we get

$$(\triangle - \frac{\partial}{\partial t})F \geq \frac{2F^2}{nt} - 2 < \nabla F, \nabla f > - \frac{F}{t}. \tag{10}$$

At a maximum of F we have $\nabla F = 0, \triangle F \leq 0$ and $F_t = 0$. Therefore

$$0 \geq \frac{2}{nt}(F^2 - \frac{nF}{2}) \tag{11}$$

Therefore $F \leq \frac{n}{2}$. Substituting in for F gives

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq \frac{n}{2t}, \tag{12}$$

which is what we wanted. $\blacksquare$

2 A Harnack inequality for a torus

Now we'll try to get a Harnack inequality out of this. Pick (x_1, t_1) and (x_2, t_2) with $t_2 \geq t_1$, and let $\eta(t) = (x_2, t_2) + t((x_1, t_1) - (x_2, t_2))$ be the straight line path from one to the other. Then

$$f(x_1, t_1) - f(x_2, t_2) = \int_0^1 \frac{df(\eta)}{ds} ds. \tag{13}$$

Calculate $\frac{df(\eta)}{ds} = \nabla f \cdot (x_1 - x_2) + f_t(t_1 - t_2)$. By inequality (12)

$$f_t(t_1 - t_2) \leq \frac{n}{2t}(t_2 - t_1) - |\nabla f|^2 (t_2 - t_1).$$

Together with (13) we get

$$f(x_1, t_1) - f(x_2, t_2) \leq \int_0^1 |\nabla f||x_2 - x_1| - |\nabla f|^2 (t_2 - t_1) + \frac{n(t_2 - t_1)}{2t} ds. \tag{14}$$

The integrand is a quadratic in $|\nabla f|^2$ with negative leading coefficient, so it has a maximum at $|\nabla f| = \frac{|x_2 - x_1|}{2(t_2 - t_1)}$, so

$$f(x_1, t_1) - f(x_2, t_2) \leq \int_0^1 \frac{|x_2 - x_1|}{2(t_2 - t_1)}|x_2 - x_1| - \left(\frac{|x_2 - x_1|}{2(t_2 - t_1)}\right)^2 (t_2 - t_1) + (t_2 - t_1)\frac{n}{2t} ds. \tag{15}$$

We split this up. for the first part

$$\int_0^1 \frac{|x_2 - x_1|}{2(t_2 - t_1)}|x_2 - x_1| - \left(\frac{|x_2 - x_1|}{2(t_2 - t_1)}\right)^2 (t_2 - t_1) ds = \frac{|x_2 - x_1|}{4(t_2 - t_1)}, \tag{16}$$

and for the second

$$\int_0^1 (t_2 - t_1)\frac{n}{2t} ds = (t_2 - t_1)\frac{n}{2}\int_0^1 \frac{1}{t_2 + s(t_1 - t_2)} ds = -\frac{n}{2}\int_{t_2}^{t_1} \frac{1}{v} dv = \frac{n}{2}\log\frac{t_2}{t_1}. \tag{17}$$

Putting these together we have

$$\log u(x_1.t_1) - \log u(x_2, x_1) \leq \frac{|x_2 - x_1|}{4(t_2 - t_1} + \frac{n}{2}\log\frac{t_2}{t_1}. \tag{18}$$

Taking exponents we get a harnack inequality,

$$\frac{u(x_1.t_1)}{u(x_2, t_2)} \leq \left(\frac{t_2}{t_1}\right)^{n/2} \exp\left(\frac{|x_2 - x_1|}{4(t_2 - t_1)}\right). \tag{19}$$