Heat equation gradinet estimate on a ball

1 generalising the proof from last time

Last time we proved a gradient estimate for functions that satisfy the heat equation on a cylinder, To do this we needed to know that the function didn't have a maximum on a boundary. We'll now generalise this proof to work on a ball B_r. To do this we will need to introduce a cutoff function. Again we will prove the case $r = 1$. Take $u : B_2 \times [0.\infty) \to \mathbb{R}$ with $\triangle u = \frac{\partial u}{\partial t}$. Let $\phi : B_2 \to \mathbb{R}$ with $\phi = 0$ on ∂B_2 and $\phi = 1$ on B_1. As before we define $f = \log u$, and $F = |\nabla f|^2 - \frac{\partial f}{\partial t}$, and try to estimate $(\triangle - \frac{\partial}{\partial t})\phi^k F$. We have

$$(\triangle - \frac{\partial}{\partial t})\phi^k F = \phi^k(\triangle - \frac{\partial}{\partial t})F + F(\triangle - \frac{\partial}{\partial t})\phi^k + 2\nabla F \cdot \nabla \phi^k. \quad (1)$$

Last time we showed that

$$(\triangle - \frac{\partial}{\partial t})F \geq 2t\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)^2 - 2 < \nabla F, \nabla f > -\frac{F}{t}, \quad (2)$$

and we can use the trick of saying

$$\sum \frac{\partial^2 f}{\partial x_i \partial x_j} \geq \frac{1}{n}\left(\sum \frac{\partial^2 f}{\partial x_i^2}\right)^2 = \frac{(\triangle f)^2}{n} = \frac{F^2}{nt^2}$$

so

$$(\triangle - \frac{\partial}{\partial t})\phi^k F \geq \phi^k\left(\frac{F^2}{nt} + t\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)^2 - 2 < \nabla F, \nabla f > -\frac{F}{t}\right) + F(\triangle - \frac{\partial}{\partial t})\phi^k + 2\nabla F \cdot \nabla \phi^k, \quad (3)$$

and then I think we're meant to use the maximum principle on $\phi^k f$ to finish it off but neither my notes nor Yaims actually states the result we're aiming for, or goes any further in the proof. can you point out mhow I'm meant to finish it? thanks.