Lecture 14: A gradient estimate for the heat equation on a ball.

1 Adapting the proof of the gradient estimate to $\mathbb{R}^n$

Last time we proved a gradient estimate for solutions of the heat equation on a torus. In this lecture we will adapt the argument to prove the same theorem on $\mathbb{R}^n$.

Theorem 1.1 *If $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is positive and satisfies the heat equation then*

$$t\left(\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u}\right) \leq \frac{n}{2}. \tag{1}$$

We will sketch the proof. We work on an interval $[0, T]$, and note that the result we want follows immediately from this . Define $f = \log u$ and $F = t\left(\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u}\right)$. Let ϕ be a cutoff function on the ball $B_1(0)$ with $0 < \phi < 1$ on the interior and $\phi = 0$ on the boundary. We can stretch this to get a cutoff function ϕ_r on $B_r(0)$ by taking $\phi_r(x) = \phi(x/r)$. If F is non positive the result is trivially true, so we can assume that $\phi_r F$ has an interior maximum without loss of generality. At this maximum

$$\triangle(\phi_r F) \leq 0, \frac{d(\phi_r F)}{dt} \geq 0, \text{ and } \phi_r \nabla F = -F \nabla \phi_r \tag{2}$$

We'll use these to get a bound on F. Calculate

$$0 \geq \left(\triangle - \frac{d}{dt}\right)(\phi_r F) \tag{3}$$

$$\geq \phi_r \triangle F + 2\nabla F \cdot \nabla \phi_r + F \triangle \phi_r - \phi_r \frac{dF}{dt} \tag{4}$$

We need to estimate some of these. The calculations are very similar to last time. We start with

$$\begin{aligned}
\triangle F &= t\triangle\left(\frac{|\nabla u|^2}{u^2}-\frac{u_t}{u}\right) && (5)\\
&= t\triangle(|\nabla f|^2 - f_t) && (6)\\
&= 2t\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)^2 + 2t\nabla(\triangle f)\cdot\nabla f - t\triangle f_t && (7)
\end{aligned}$$

by Bochner. Calculate $\triangle f = \frac{\partial}{\partial x_i}\frac{\partial u/\partial x_i}{u} = \frac{\triangle u}{u} - \frac{|\nabla u|^2}{u^2} = -\frac{F}{t}$ to get

$$\triangle F = 2t\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)^2 - 2\nabla F\cdot\nabla f - t\triangle f_t. \tag{8}$$

Recall the inequality $(\sum A_i i)^2 \leq n\sum(A_{ii}^2)$ for all matrices A from last time, and apply it to the hessian of f to give

$$\begin{aligned}
\triangle F &= \frac{2t}{n}(\triangle f)^2 - 2\nabla F\cdot\nabla f - t\triangle f_t && (9)\\
&\geq \frac{2F^2}{nt} - 2\nabla F\cdot\nabla f - \triangle f_t. && (10)
\end{aligned}$$

We also need an estimate on F_t. We have

$$F_t = |\nabla f|^2 - f_t + t(2\nabla f\cdot\nabla f_t) - tf_{tt}, \tag{11}$$

and $\triangle f + |\nabla f|^2 = f_t$, so

$$\begin{aligned}
F_t &= |\nabla f|^2 - f_t + t(2\nabla f\cdot\nabla f_t) - t(\triangle f + |\nabla f|^2)_t && (12)\\
&= \frac{F}{t} - t\triangle f_t. && (13)
\end{aligned}$$

Putting $4, 10$ and 13 together we get

$$0 \geq \phi_r\left(\frac{2F^2}{nt} - 2\nabla F\cdot\nabla f - \frac{F}{t}\right) + 2\nabla F\cdot\nabla\phi_r + F\triangle\phi_r. \tag{14}$$

Recall that $\phi_r\nabla F = -F\nabla\phi_r$, so

$$\begin{aligned}
0 &\geq \phi_r\left(\frac{2F^2}{nt} + \frac{2F}{\phi_r}\nabla\phi_r\cdot\nabla f - \frac{F}{t}\right) - \frac{2F}{\phi_r}|\nabla\phi_r| + F\triangle\phi_r && (15)\\
&\geq F\phi_r\left(\frac{2F}{nt} + \frac{2}{\phi_r}\nabla\phi_r\cdot\nabla f - \frac{1}{t} - 2\frac{|\nabla\phi_r|^2}{\phi_r^2} + \frac{\triangle\phi_r}{\phi_r}\right). && (16)
\end{aligned}$$

Now use an absorbing inequality $\frac{\partial \phi_r}{\partial x_i}\frac{\partial f}{\partial x_i} \geq -\frac{1}{\epsilon}\left(\frac{\partial \phi_r}{\partial x_i}\right)^2 - \epsilon\left(\frac{\partial f}{\partial x_i}\right)^2$ for all $\epsilon > 0$. to show that

$$\nabla \phi_r \cdot \nabla f \geq -\frac{1}{\epsilon}|\nabla \phi_r|^2 - \epsilon|\nabla f|^2 \tag{17}$$

for all $\epsilon > 0$. Consequently

$$0 \geq F\phi_r \left(\frac{2F}{nt} - \frac{2}{\phi_r}\left(\frac{1}{\epsilon}|\nabla \phi_r|^2 + \epsilon|\nabla f|^2\right) - \frac{1}{t} - 2\frac{|\nabla \phi_r|^2}{\phi_r^2} + \frac{\triangle \phi_r}{\phi_r}\right). \tag{18}$$

Let $r \to \infty$ so that $|\nabla \phi_r|$ and $\triangle \phi_r$ tend to zero and $\phi_r \to 1$, and get

$$0 \geq F\left(\frac{2F}{nt} - 2\epsilon|\nabla f|^2 - \frac{1}{t}\right). \tag{19}$$

Finally we let $\epsilon \to 0$ and recover

$$0 \geq \frac{F}{t}\left(\frac{2F}{n} - 1\right). \tag{20}$$

From this we get$F \leq n/2$ as required.