# The decay of solutions of the heat equation, Campanato's lemma, and Morrey's Lemma

## 1 The decay of solutions of the heat equation

A few lectures ago we introduced the heat equation

$$\triangle u = u_t \tag{1}$$

for functions of both space and time. we will now bound the decay of the Dirichlet energy and the $L_2$ norm (ie $\int u^2$) of solutions. If $u$ solves the heat equation on $\Omega \subset \mathbb{R}^n$ with $u = 0$ on the boundary then

$$\frac{d}{dt}\int_\Omega u^2 = 2\int_\Omega u u_t = 2\int_\Omega u\triangle u = -2\int_\Omega |\nabla u|^2, \tag{2}$$

and

$$\frac{d}{dt}\int_\Omega |\nabla u|^2 = 2\int_\Omega \nabla u\cdot\nabla u_t = -2\int_\Omega u_t\triangle u = -2\int_\Omega |\triangle u|^2. \tag{3}$$

Also note that the dirichlet energy is decreasing, so

$$-2\int_\Omega |\nabla u(\cdot,0)|^2 \le \frac{d}{dt}\int_\Omega u^2 \le -2\int_\Omega |\nabla u(\cdot,t)|^2, \tag{4}$$

and

$$-2T\int_\Omega |\nabla u(\cdot,0)|^2 \le \int_\Omega u^2(\cdot,T) - \int_\Omega u^2(\cdot,0) \le -2T\int_\Omega |\nabla u(\cdot,T)|^2. \tag{5}$$

## 2 Campanato's Lemma

Before discussing Campanato's lemma we need the concept of Holder continuity.

**Definition** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. $f$ is $\alpha$ continuous at $x_0$ if there are constants $k$ and $0 < \alpha \le 1$ such that

$$f(x_0 + x) - f(x_0) \le k|x|^\alpha \tag{6}$$

for small $x_0$. If $f$ is everywhere Holder continuous for some constant $\alpha$ then we say that $f \in C^\alpha$.

Now take $f$ continuous on $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$, and define

$$A_{x,r} = \frac{1}{\text{vol } B_r(x)} \int_{B_r(x)} u. \tag{7}$$

Campanato's Lemma is then

**Lemma 2.1** *(Campanato's Lemma) If there is a constant $c$ such that*

$$\int_{B_r(x)} (f - A_{x,r})^2 \leq c r^{2\alpha+n} \tag{8}$$

*for all $x$ and for small $r$ then $f \in C^\alpha$.*

**Proof** First we will bound $|A_{x,r} - A_{x,2r}|$ Calculate

$$|A_{x,r} - A_{x,2r}|^2 = \left( \frac{1}{\text{vol } B_r(x)} \int_{B_r(x)} f - \frac{1}{\text{vol } B_{2r}(x)} \int_{B_{2r}(x)} f \right)^2 \tag{9}$$

$$= \left( \frac{1}{\text{vol } B_r(x)} \int_{B_r(x)} \left( f - \frac{1}{\text{vol } B_{2r}(x)} \int_{B_{2r}(x)} f \right) \right)^2 \tag{10}$$

$$\leq \left( \frac{1}{\text{vol } B_r(x)} \int_{B_r(x)} |f - A_{x,2r}| \right)^2 \tag{11}$$

$$\leq \left( \frac{2^n}{\text{vol } B_{2r}(x)} \int_{B_{2r}(x)} |f - A_{x,2r}| \right)^2. \tag{12}$$

By Cauchy-Schwarz $\left(\int gh\right)^2 \leq \left(\int g^2\right) \left(\int h^2\right)$. Apply this with $g = 1$ and $h = f - A_{x,2r}$ to get

$$|A_{x,r} - A_{x,2r}|^2 \leq \frac{2^{2n}}{\text{vol } B_{2r}(x)} \int_{B_{2r}(x)} (f - A_{x,2r})^2. \tag{13}$$

Now use the condition, 8, so

$$|A_{x,r} - A_{x,2r}|^2 \leq \frac{2^{2n} c (2r)^{2\alpha+n}}{\text{vol } B_{2r}(x)}, \tag{14}$$

and we can pick a new dimensional constant $C = \sqrt{\frac{c 2^{2n+2\alpha}}{\text{vol } B_1(x)}}$ so that

$$|A_{x,r} - A_{x,2r}| \leq C r^\alpha. \tag{15}$$

We apply this to our problem. Note that $A_{x,2^{-k}r} - A_{x,r} = \sum_{i=0}^{i=k-1} A_{x,2^{-i-1}r} - A_{x,2^{-i}r}$, so

$$\begin{aligned} |A_{x,2^{-k}r} - A_{x,r}| &\leq \sum_{i=0}^{i=k-1} |A_{x,2^{-i-1}r} - A_{x,2^{-i}r}| && (16)\\ &\leq \sum_{i=0}^{i=k-1} C(2^{-i-1}r)^{\alpha} && (17)\\ &\leq Cr^{\alpha}2^{-\alpha} \sum_{i=0}^{i=k-1} 2^{-i\alpha}. && (18) \end{aligned}$$

This is simply the sum of a geometric series, so we can use the usual formula to get

$$|A_{x,2^{-k}r} - A_{x,r}| \leq \frac{1-2^{-k\alpha}}{1-2^{-\alpha}} Cr^{\alpha}2^{-\alpha}. \tag{19}$$

Now let $k \to \infty$ to get

$$|f(x) - A_{x,r}| \leq \frac{1}{1-2^{-\alpha}} Cr^{\alpha}2^{-\alpha}. \tag{20}$$

Now pick another point $y \in \mathbb{R}^n$. Clearly we also have $|f(y) - A_{y,s}| \leq \frac{1}{1-2^{-\alpha}} Cs^{\alpha}2^{-\alpha}$. Now we'll estimate $|A_{x,r} - A_{y,s}|$. Calculate

$$\begin{aligned} |A_{x,r} - A_{y,s}|^2 &= \left| \frac{1}{\text{vol } B_r(x)} \int_{B_r(x)} f - A_{y,s} \right|^2 && (21)\\ &= \left| \frac{1}{\text{vol } B_r(x)} \int_{B_r(x)} (f - A_{y,s}) \right|^2 && (22)\\ &\leq \left( \frac{1}{\text{vol } B_r(x)} \int_{B_r(x)} |f - A_{y,s}| \right)^2, && (23) \end{aligned}$$

and apply Cauchy-Schwaz as before to get

$$|A_{x,r} - A_{y,s}|^2 \leq \frac{1}{\text{vol } B_r(x)} \int_{B_r(x)} (f - A_{y,s})^2 \tag{24}$$

Pick $r = |x-y|$ and $s = 2|x-y|$ so that $B_r(x) \subset B_s(y)$. Therefore

$$\begin{aligned} |A_{x,r} - A_{y,s}|^2 &\leq \frac{1}{\text{vol } B_r(x)} \int_{B_s(y)} (f - A_{y,s})^2 && (25)\\ &\leq \frac{\text{vol } B_s(y)}{\text{vol } B_r(x)} \frac{1}{\text{vol } B_s(y)} \int_{B_s(y)} (f - A_{y,s})^2 && (26) \end{aligned}$$

and, by our hypothesis,

$$|A_{x,r} - A_{y,s}|^2 \leq \frac{\text{vol } B_s(y)}{\text{vol } B_r(x)} cs^{2\alpha+n}. \tag{27}$$

Plugging in our values for $r$ and $s$ we have

$$|A_{x,r} - A_{y,s}|^2 \leq 2^n c(2|x-y|)^{2\alpha+n}, \tag{28}$$

and we're only interested in small $|x-y|$, so we can take $2|x-y| \leq 1$ to get

$$|A_{x,r} - A_{y,s}|^2 \leq 2^n c(2|x-y|)^{2\alpha}. \tag{29}$$

At last we can calculate

$$\begin{aligned}
|f(x) - f(y)| &\leq |f(x) - A_{x,r}| + |A_{x,r} - A_{y,s}| + |A_{y,s} - f(y)| && (30)\\
&\leq \frac{1}{1-2^{-\alpha}} Cr^\alpha 2^{-\alpha} + 2^{n/2}\sqrt{c}(2|x-y|)^\alpha + \frac{1}{1-2^{-\alpha}} Cs^\alpha 2^{-\alpha} && (31)\\
&\leq \left( \frac{C2^{-\alpha}}{1-2^{-\alpha}} + 2^{n/2+\alpha}\sqrt{c} + \frac{1}{1-2^{-\alpha}} C \right) |x-y|^\alpha. && (32)
\end{aligned}$$

This completes the proof. ∎

# 3 Morrey's Lemma

Morrey's lemma is very similar to Campanato's lemma, but uses a condition on Dirichlet energy rather than $L_2$ norm.

**Lemma 3.1** *(Morrey's lemma) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous. If there is a constant $c_1$ with*

$$\int_{B_r(x)} |\nabla f|^2 \leq c_1 r^{n-2-2\alpha} \tag{33}$$

*for all $x$ and small $r$ then $f \in C^\alpha$.*

**Proof** This is a straightforward consequence of Campanato and Poincare. By Poincare there is a constant $c_2$ such that

$$\int_{B_r(x)} (f - A_{x,r})^2 \leq c_2 r^2 \int_{B_r(x)} |\nabla f|^2 \leq c_1 c_2 r^{n+2\alpha} \tag{34}$$

by hypothesis. The result then follows by Campanato.