Five inequalities for Harmonic functions

In this lecture we will prove five inequalities for harmonic functions.

1 Bounding integrals of Harmonic functions

Proposition 1.1 *Let r and s be real numbers with $0 < r \leq s$, and $x \in \mathbb{R}^n$. There are constants c_i such that*

$$\int_{B_r(x)} f^2 \leq c_1 \left(\frac{r}{s}\right)^n \int_{B_s(x)} f^2, \tag{1}$$

$$\int_{B_r(x)} (f - A_{x,r})^2 \leq c_2 \left(\frac{r}{s}\right)^{n+2} \int_{B_s(x)} (f - A_{x,s})^2, \tag{2}$$

$$\int_{B_r(x)} |\nabla f|^2 \leq c_3 \left(\frac{r}{s}\right)^n \int_{B_s(x)} |\nabla f|^2, \tag{3}$$

and

$$\int_{B_r(x)} |\nabla f - (\nabla f)_{x,r}|^2 \leq c_4 \left(\frac{r}{s}\right)^{n+2} \int_{B_s(x)} |\nabla f - (\nabla f)_{x,s}|^2. \tag{4}$$

for all functions f that are harmonic on $B_s(x)$ with $A_{x,t}, (\nabla f)_{x,t}$ the averages of f and ∇f over $B_t(x)$ respectively.

Before proving these we will prove another inequality, the mean value inequality.

Proposition 1.2 *If f is harmonic on $B_{2r}(x)$ then*

$$\sup_{B_r(x)} f^2 \leq \frac{2^n}{vol\ B_{2r}(x)} \int_{B_r(x)} f^2. \tag{5}$$

Proof Pick $y \in B_r(x)$. By the mean value property (from lecture 1)

$$f(y) = \frac{1}{\text{vol } B_r(y)} \int_{B_r(y)} f, \tag{6}$$

so

$$
\begin{aligned}
f^2(y) &= \left(\frac{1}{\text{vol } B_r(y)} \int_{B_r(y)} f \right)^2 & (7)\\
&= \left(\frac{1}{\text{vol } B_r(y)} \right)^2 \left(\int_{B_r(y)} f \right)^2 & (8)\\
&\leq \left(\frac{1}{\text{vol } B_r(y)} \right)^2 \left(\int_{B_r(y)} f^2 \right) \left(\int_{B_r(y)} 1^2 \right) & (9)\\
&\leq \frac{1}{\text{vol } B_r(y)} \int_{B_r(y)} f^2 & (10)
\end{aligned}
$$

by Cauchy Schwarz. Note that $B_r(y) \subset B_{2r}(x)$, so we can expand the area of integration to get

$$
\begin{aligned}
f^2(y) &\leq \frac{1}{\text{vol } B_r(y)} \int_{B_{2r}(x)} f^2 & (11)\\
&\leq \frac{2^n}{\text{vol } B_{2r}(x)} \int_{B_{2r}(x)} f^2. & (12)
\end{aligned}
$$

Therefore

$$
\sup_{B_r(x)} f^2 \leq \frac{2^n}{\text{vol } B_{2r}(x)} \int_{B_{2r}(x)} f^2 \qquad (13)
$$

as required. ∎

Now we'll use this to get our first inequality. If $r \leq s \leq 2r$ then

$$
\begin{aligned}
\int_{B_r(x)} f^2 \leq \int_{B_s(x)} f^2 & & (14)\\
&\leq \left(\frac{2r}{s} \right)^n \int_{B_s(x)} f^2 & (15)\\
&\leq 2^n \left(\frac{r}{s} \right)^n \int_{B_s(x)} f^2. & (16)
\end{aligned}
$$

If instead $2r \leq s$ then

$$
\begin{aligned}
\sup_{B_r(x)} f^2 &\leq \sup_{B_{s/2}(x)} f^2 & (17)\\
&\leq \frac{2^n}{\text{vol } B_s(x)} \int_{B_s(x)} f^2 & (18)
\end{aligned}
$$

by the mean value inequality Therefore

$$\frac{1}{\text{vol } B_r(x)} \int_{B_r(x)} f^2 \leq \frac{1}{\text{vol } B_r(x)} \int_{B_r(x)} \left(\frac{2^n}{\text{vol } B_s(x)} \int_{B_s(x)} f^2 \right) \tag{19}$$

$$\leq \frac{2^n}{\text{vol } B_s(x)} \int_{B_s(x)} f^2, \tag{20}$$

and the ration of the volumes is $\left(\frac{r}{s}\right)^n$, so

$$\int_{B_r(x)} f^2 \leq 2^n \left(\frac{r}{s}\right)^n \int_{B_s(x)} f^2 \tag{21}$$

for large s as well.

Note that $\triangle \frac{\partial f}{\partial x_i} = \frac{\partial}{\partial x_i} \triangle f = 0$. Therefore 3 follows immediately from 1. Now we'll prove 2. First consider the case $4r \leq s$. Since $\frac{\partial f}{\partial x_i}$ is harmonic we can apply the mean value inequality to get

$$\sup_{B_r(x)} |\nabla f|^2 \leq \frac{1}{\text{vol } B_{2r}(x)} \int_{B_{2r}(x)} |\nabla f|^2. \tag{22}$$

Now apply this. By the intermediate value theorem there is $y \in B_r(x)$ with $f(y) = A_{x,r}$. Pick $z \in B_r(x)$. Clearly $|f(z) - f(y)| \leq |z - y| \sup_{B_r(x)} |\nabla f| \leq 2r \sup_{B_r(x)} |\nabla f|$. Therefore

$$\frac{1}{\text{vol } B_r(x)} \int_{B_r(x)} (f - A_{x,r})^2 \leq \frac{1}{\text{vol } B_r(x)} \int_{B_r(x)} \left(2r \sup_{B_r(x)} |\nabla f| \right)^2 \tag{23}$$

$$\leq 4r^2 \sup_{B_r(x)} |\nabla f|^2 \tag{24}$$

$$\leq 4r^2 \sup_{B_{s/4}(x)} |\nabla f|^2 \tag{25}$$

$$\leq 4r^2 \frac{1}{\text{vol } B_{s/2}(x)} \int_{B_{s/2}(x)} |\nabla f|^2. \tag{26}$$

Apply Cacciopolli to get $\int_{B_{s/2}(x)} |\nabla f|^2 \leq \frac{1}{s^2} \int_{B_s(x)} (f - A_{x,s})^2$, so

$$\frac{1}{\text{vol } B_r(x)} \int_{B_r(x)} (f - A_{x,r})^2 \leq \frac{4r^2}{s^2 \text{vol } B_{s/2}(x)} \int_{B_s(x)} (f - A_{x,s})^2, \tag{27}$$

and

$$\int_{B_r(x)} (f - A_{x,r})^2 \leq 2^{n+2} \left(\frac{r}{s}\right)^{n+2} \int_{B_s(x)} (f - A_{x,s})^2 \tag{28}$$

as required. For $r \leq s \leq 4r$ we simply note that

$$\int_{B_r(x)} (f - A_{x,r})^2 \leq 4^{n+2} \left(\frac{r}{s}\right)^{n+2} \int_{B_r(x)} (f - A_{x,r})^2 \ \leq 4^{n+2} \left(\frac{r}{s}\right)^{n+2} \int_{B_s(x)} (f - A_{x,r})^2. \quad (29)$$

This completes the proof of 2. The final inequality, 4, follows from 2 in exactly the same way that 3 follows from 1.

We can also prove 1, 2, 3, and 4 for L harmonic operators when L is a uniformly elliptic operator taking $Lu = A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}$. In this case the constants c_i depend on the operator. Proofs are omitted.