Lecture 18: Regularity of L harmonic functions Part III

1 Finishing the proof

In order to finish the proof from last time we need a lemma.

Lemma 1.1 *Let ϕ be a positive and increasing function on the positive reals, and let α, c be positive constants. For all $0 < \gamma < \alpha$ there is $\epsilon > 0$ such that*

$$\phi(r) \leq c\left(\left(\frac{r}{s}\right)^{\alpha} + \epsilon\right)\phi(s) \tag{1}$$

for $0 < r < s$ implies

$$\phi(r) \leq c'\left(\frac{r}{s}\right)^{\gamma}\phi(s) \tag{2}$$

for some constant c'.

Proof Choose $0 < \tau < 1$ such that $\epsilon \leq \tau^{\alpha}$. Then

$$\phi(\tau s) \leq c(\tau^{\alpha} + \epsilon)\phi(s) \leq 2c\tau^{\alpha}\phi(s). \tag{3}$$

Therefore

$$\phi(\tau^k s) \leq (2c\tau^{\alpha})^k \phi(s). \tag{4}$$

Pick γ so that $2c\tau^{\alpha-\gamma} \leq 1$ and we have

$$\phi(\tau^k s) \leq \tau^{k\gamma}\phi(s). \tag{5}$$

When $r = \tau^k s$ this is precisely what we wanted with $c' = 1$. If instead $\tau^{k+1}s \leq r \leq \tau^k s$ then

$$\phi(r) \leq \phi(\tau^k s) \leq \tau^{k\gamma}\phi(s) \leq \frac{1}{\tau}\left(\frac{r}{s}\right)^{\gamma}\phi(s) \tag{6}$$

which is what we needed. Note that by taking τ very small we can get γ as close to α as we like. ∎

Now we apply this to what we were doing last time. Let $\phi(r) = \int_{B_r(x_0)} |\nabla u|^2$. We showed that

$$\phi(r) \leq \left(\left(\frac{r}{s} \right)^n + k||A_{ij} - A_{ij}(x_0)|| \right) \phi(s) \tag{7}$$

By picking s we can get $||A_{ij} - A_{ij}(x_0)||$ as small as we like so we will be able to apply our lemma. Pick $0 < \beta < 1$ and set $\gamma = n - 2 + 2\beta$. By our lemma there is a constant k' with

$$\phi(r) \leq k' \left(\frac{r}{s} \right)^{n-2+2\beta} \phi(s). \tag{8}$$

In our old notation this is

$$\int_{B_r(x_0)} |\nabla u|^2 \leq k' \left(\frac{r}{s} \right)^{n-2+2\beta} \int_{B_s(x_0)} |\nabla u|^2, \tag{9}$$

and Holder continuity follows by Morrey's lemma.