# Lecture Two: The Gradient Estimate

## 1 The Bochner Formula

For an $m \times m$ real matrix we can define a norm by taking

$$|A|^2 = \sum_{i,j} a_{ij}^2.$$

In particular, if $\Omega$ is some subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$, then we can take the norm of the Hessian.

$$|\text{Hess } u|^2 = \sum_{i,j} \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2. \tag{1}$$

Using this we can prove the following.

**Proposition 1.1** *(The Bochner formula). Let $u$ be a real valued function on some open subset of $\mathbb{R}^n$, then*

$$\frac{1}{2} \triangle |\nabla u|^2 = < \nabla \triangle u, \nabla u > + |\textit{Hess } u|^2, \tag{2}$$

*where $< x, y >$ indicates the usual dot product of $x$ and $y$.*

**Proof** Proof is by calculation

$$\begin{aligned}
\frac{1}{2} \triangle |\nabla u|^2 &= \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i^2} \left( \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_j} \right) \\
&= \sum_{i,j} \frac{\partial}{\partial x_i} \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_j} \right) \\
&= \sum_{i,j} \frac{\partial^3 u}{\partial x_i^2 \partial x_j} \frac{\partial u}{\partial x_j} + \sum_{i,j} \frac{\partial^2 u}{\partial x_i \partial_j} \frac{\partial^2 u}{\partial x_i \partial_j} \\
&= \sum_{j} \frac{\partial}{\partial x_j} (\triangle u) \frac{\partial u}{\partial x_j} + |\text{Hess } u|^2 \\
&= < \nabla \triangle u, \nabla u > + |\text{Hess } u|^2. \quad \blacksquare
\end{aligned}$$

## 2 The Gradient Estimate

We now prove a gradient estimate for harmonic functions.

**Theorem 2.1** *There are dimensional constants $c(n)$ such that*

$$\sup_{B_r(x_0)} |\nabla u| \leq \frac{c(n)}{r} \sup_{B_{2r}(x_0)} |u|. \tag{3}$$

*for all harmonic functions $u$ on $B_{2r}(x_0) \subset \mathbb{R}^n$*

**Proof** Note that it suffices to check the case $x_0 = 0$. Now proceed as follows.

**Step 1.** Show that a sub-harmonic function on a ball takes it's maximum on the boundary. Let $p : B_{2r}(x_0) \to \mathbb{R}$ be sub-harmonic. Note that $\triangle|x|^2 = 2n$, so $\triangle(p+\epsilon|x|^2) > 0$ for all $\epsilon > 0$, Therefore, by the maximum principle, $p + \epsilon|x|^2$ has no interior maximum, so it's maximum occurs on the boundary. Letting $\epsilon \to 0$ we see that $p$ takes its maximum on the boundary as well.

**Step 2.** Prove the result for $r = 1$. Take $u$ harmonic on $B_2(0)$, and introduce a test function $\phi$ with $\phi = 0$ on the boundary of $B_2(0)$ and $\phi > 0$ on the interior. We will work with $\triangle(\phi^2|\nabla u|^2)$, and apply Bochner to simplify. Calculate

$$\begin{aligned}
\triangle(\phi^2|\nabla u|^2) &= \phi^2\triangle|\nabla u|^2 + |\nabla u|^2\triangle(\phi^2) + 2\nabla(\phi^2)\cdot\nabla(|\nabla u|^2) \\
&= 2\phi^2|\text{Hess } u|^2 + 2\phi^2 < \nabla\triangle u, \nabla u > + |\nabla u|^2\triangle(\phi^2) + 8\sum_{i,j}\phi\frac{\partial\phi}{\partial x_i}\frac{\partial^2 u}{\partial x_j x_i}\frac{\partial u}{\partial x_j} \\
&= 2\phi^2|\text{Hess } u|^2 + |\nabla u|^2\triangle(\phi^2) + 8\sum_{i,j}\phi\frac{\partial\phi}{\partial x_i}\frac{\partial^2 u}{\partial x_j x_i}\frac{\partial u}{\partial x_j}.
\end{aligned}$$

Define $a_{ij} = \phi\frac{\partial^2 u}{\partial x_i \partial x_j}$ and $b_{ij} = \frac{\partial\phi}{\partial x_i}\frac{\partial u}{\partial x_j}$. We can re-write to get

$$\triangle(\phi^2|\nabla u|^2) = |\nabla u|^2\triangle(\phi^2) + 2\sum_{i,j}\left(a_{ij}^2 + 4a_{ij}b_{ij}\right). \tag{4}$$

Note that $a_{ij}^2 + 4a_{ij}b_{ij} + 4b_{ij}^2 = (a_{ij} + 2b_{ij})^2 \geq 0$. Apply this to (4) to give

$$\triangle(\phi^2|\nabla u|^2) \geq |\nabla u|^2\triangle(\phi^2) - 8\sum_{ij} b_{ij}^2 \tag{5}$$

or, in our original notation,

$$\triangle(\phi^2|\nabla u|^2) \geq -8\sum_{ij}\left(\frac{\partial\phi}{\partial x_i}\right)^2\left(\frac{\partial u}{\partial x_j}\right)^2 + |\nabla u|^2\triangle(\phi^2) \quad (6)$$

$$\geq -8|\nabla\phi|^2|\nabla u|^2 + \triangle(\phi^2)]|\nabla u|^2. \quad (7)$$

Observe that $\triangle(u^2) = 2u\triangle u + 2|\nabla u|^2 = 2|\nabla u|^2$ since $\triangle u = 0$. Let $k(n) = |\inf_{B_2(0)}(-8|\nabla\phi|^2 + \triangle(\phi^2))|$, so

$$\triangle(\phi^2|\nabla u|^2 + k(n)u^2) \geq 0 \text{ on } B_2(0). \quad (8)$$

By step 1 $\sup_{B_2(0)}(\phi^2|\nabla u|^2 + k(n)u^2)$ occurs on the boundary. Furthermore $\phi$ vanishes on the boundary, so we get

$$\sup_{\delta B_2(0)} k(n)u^2 \geq \sup_{B_1(0)} \phi^2|\nabla u|^2. \quad (9)$$

Let $h(n) = \inf_{B_1(0)}\phi > 0$ and rearrange to give

$$\frac{k(n)}{h(n)^2}\sup_{\delta B_2(0)}u^2 \geq \sup_{B_1(0)}|\nabla u|^2. \quad (10)$$

Finally, take square roots to get

$$\sup_{B_1(0)}|\nabla u| \leq c(n)\sup_{B_2(0)}|u| \quad (11)$$

as required, with $c(n) = \left(\frac{k(n)}{h(n)^2}\right)^{1/2}$. It is important to note that $c(n)$ really is a dimensional constant: although it depends on a choice of $\phi$ it doesn't depend on $u$.

**Step 3.** Extend this to general $r$. If $u$ is harmonic on $B_{2r}(0)$ define $\tilde{u}(x) = u(x/r)$, and note that $\tilde{u}$ is harmonic on $B_2(0)$. Therefore, by (11),

$$\sup_{B_1(x_0)}|\nabla\tilde{u}| \leq c(n)\sup_{B_2(0)}|\tilde{u}|, \quad (12)$$

so

$$\sup_{B_r(0)} r|\nabla u| \leq c(n)\sup_{B_{2r}(0)}|u|. \quad (13)$$

This completes the proof. ∎