# Lecture 21: The mean value inequality for uniformly elliptic operators part I

## 1 The mean value inequality: Iterative argument

In this lecture we will prove a mean value inequality for uniformy elliptic operators in divergence form . The argument is an iterative one due to De Georgi, Nash, and Moser. As usual we take $L$ an operator with

$$Lu = \frac{\partial}{\partial x_i} A_{ij} \frac{\partial u}{\partial x_j} \tag{1}$$

and $\lambda |v|^2 \leq A_{ij} v_i v_j \leq \Lambda |v|^2$ for all vectors $v$. Let $u$ be a function satisfying $u \geq 0, Lu \geq 0$. Take $x_0$ a point, and $R$ a fixed positive number. Let $\phi$ be a test function on $B_R(x_0)$ which is zero on the boundary. Clearly

$$\int_{B_R(x_0)} \phi^2 u A \nabla u \cdot dS = 0 \tag{2}$$

so, by Stokes' theorem,

$$\int_{B_R(x_0)} \phi^2 u L u + \int_{B_R(x_0)} A_{ij} \frac{\partial \phi^2 u}{\partial x_i} \frac{\partial u}{\partial x_j} = 0 \tag{3}$$

and, since the first term is non-negative,

$$0 \geq \int_{B_R(x_0)} A_{ij} \frac{\partial \phi^2 u}{\partial x_i} \frac{\partial u}{\partial x_j}. \tag{4}$$

We can simplify this a bit to get

$$0 \geq \int_{B_R(x_0)} A_{ij} \phi^2 \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + 2 \int_{B_R(x_0)} A_{ij} \phi u \frac{\partial \phi}{\partial x_i} \frac{\partial u}{\partial x_j}. \tag{5}$$

and

$$-2 \int_{B_R(x_0)} A_{ij} \phi u \frac{\partial \phi}{\partial x_i} \frac{\partial u}{\partial x_j} \geq \int_{B_R(x_0)} A_{ij} \phi^2 \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}. \tag{6}$$

Apply uniform ellipticity to the right hand side to get

$$\lambda \int_{B_R(x_0)} \phi^2 |\nabla u|^2 \leq -2 \int_{B_R(x_0)} A_{ij} \phi u \frac{\partial \phi}{\partial x_i} \frac{\partial u}{\partial x_j}. \tag{7}$$

Now work on the other term. At each point the matrix $A$ defines a good metric, so Cauchy-Schwarz applies, and we get $-\phi u A_{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial u}{\partial x_j} \leq \phi u \left(\nabla \phi \cdot A \nabla \phi\right)^{1/2} \left(\nabla u \cdot A \nabla \phi u\right)^{1/2}$, so

$$\lambda \int_{B_R(x_0)} \phi^2 |\nabla u|^2 \leq 2 \int_{B_R(x_0)} \phi u \left(\nabla \phi \cdot A \nabla \phi\right)^{1/2} \left(\nabla u \cdot A \nabla \phi u\right)^{1/2}. \tag{8}$$

Use Cauchy-Schwarz again in the form $\int fg \leq \left(\int f^2)\right)^{1/2} \left(\int g^2)\right)^{1/2}$ to get

$$\lambda \int_{B_R(x_0)} \phi^2 |\nabla u|^2 \leq 2 \left( \int_{B_R(x_0)} u^2 \nabla \phi \cdot A \nabla \phi \right)^{1/2} \left( \int_{B_R(x_0)} \phi^2 \nabla u \cdot A \nabla u \right)^{1/2}. \tag{9}$$

Uniform ellipticity then gives

$$\lambda \int_{B_R(x_0)} \phi^2 |\nabla u|^2 \leq 2\Lambda \left( \int_{B_R(x_0)} u^2 |\nabla \phi|^2 \right)^{1/2} \left( \int_{B_R(x_0)} \phi^2 |\nabla u|^2 \right)^{1/2}, \tag{10}$$

so rearrange to get

$$\int_{B_R(x_0)} \phi^2 |\nabla u|^2 \leq \frac{4\Lambda^2}{\lambda^2} \int_{B_R(x_0)} u^2 |\nabla \phi|^2. \tag{11}$$

This should be familiar, as we proved it on the way to the Cacciopolli inequality in lecture 6. We'll apply it slightly differently this time. Consider

$$\int_{B_R(x_0)} |\nabla(\phi u)|^2 = \int_{B_R(x_0)} |\phi \nabla u + u \nabla \phi|^2 \tag{12}$$

$$\leq 2 \int_{B_R(x_0)} \phi^2 |\nabla u|^2 + 2 \int_{B_R(x_0)} u^2 |\nabla \phi|^2. \tag{13}$$

Combining this with 11 we get

$$\int_{B_R(x_0)} |\nabla(\phi u)|^2 \leq k \int_{B_R(x_0)} u^2 |\nabla \phi|^2 \tag{14}$$

for a constant $k = 2 + \frac{8\Lambda^2}{\lambda^2}$. Now we need to use the Sobolev inequality. For Simplicity we will assume that $n \geq 3$, but a similar result holds in the other cases.

**Theorem 1.1** *Let $\Omega \subset \mathbb{R}^n$ with $n \geq 3$, and let $w$ be a function with compact support on $\Omega$. Then*

$$\left(\int_\Omega |w|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq c \int_\Omega |\nabla w|^2. \tag{15}$$

We won't prove this here. Apply it with $w = \phi u$ (this has compact support because $\phi$ does) to get

$$\int_{B_R(x_0)} (\phi u)^{\frac{2n}{n-2}} \leq c \int_{B_R(x_0)} |\nabla(\phi u)|^2 \leq \tilde{c} \int_{B_R(x_0)} u^2 |\nabla \phi|^2. \tag{16}$$

for some constant $\tilde{c}$.

Define $A_{r,k} = B_r(x_0) \cap \{u > k\}$, and let $|A_{r,k}|$ be the volume of this set. For any function $f$ define $f_+$ to be the positive part, i.e.

$$f_+ = \sup(f, 0). \tag{17}$$

Note that if $u$ is $L$ harmonic then $u_+$ is $L$ harmonic almost everywhere, and claim without proof that everything we've done today goes through for the positive part of a harmonic function as well as for completely harmonic functions. Also pick $r < R$, and set

$$\phi = \begin{cases} 1 & \text{on } B_r(x_0) \\ \frac{R-|x|}{R-r} & \text{on } B_R(x_0) \setminus B_r(x_0), \text{ and} \\ 0 & \text{outside } B_R(x_0) \end{cases} \tag{18}$$

so that $|\nabla \phi| = \frac{1}{R-r}$ on $B_R(x_0)$, and 0 elsewhere. Note that if $u$ is $L$-harmonic then $u - k$ is also $L$ harmonic. Putting all this together we get

$$\left(\int_{A_{r,k}} |(u-k)_+|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq \left(\int_{B_R(x_0)} |\phi(u-k)_+|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \tag{19}$$

$$\leq \tilde{c} \int_{B_R(x_0)} |\nabla \phi|^2 ((u-k)_+)^2 \tag{20}$$

$$\leq \frac{\tilde{c}}{(R-r)^2} \int_{A_{R,k} \setminus B_r(x_0)} ((u-k)_+)^2. \tag{21}$$

Now we'll introduce another important inequality: the Holder Inequality.

**Theorem 1.2** *Let $f, g$ be functions, and $p, q$ real numbers satisfying $\frac{!}{p} + \frac{1}{q} = 1$. Then*

$$\int fg \leq \left(\int f^p\right)^{1/p} \left(\int g^q\right)^{1/q}. \tag{22}$$

This is simply a generalisation of the Cauchy-Schwarz inequality, which is the case $p = q = 2$. Apply this with $p = \frac{n}{n-2}, q = \frac{n}{2}$ and any function $f$ on any set $\Omega$ to get

$$\int_\Omega f^2 \leq \left(\int_\Omega f^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} |\Omega|^{\frac{2}{n}}. \tag{23}$$

Set $f = (u-k)_+$ and $\Omega = A_{r,k}$ and we get

$$\begin{aligned}
\int_{A_{r,k}} ((u-k)_+)^2 &\leq \left(\int_{A_{r,k}} ((u-k)_+)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} |A_{r,k}|^{\frac{2}{n}} && (24)\\
&\leq \frac{\tilde{c}|A_{r,k}|^{\frac{2}{n}}}{(R-r)^2} \int_{A_{R,k}\setminus B_r(x_0)} ((u-k)_+)^2 && (25)\\
&\leq \frac{\tilde{c}|A_{r,k}|^{\frac{2}{n}}}{(R-r)^2} \int_{A_{R,k}} ((u-k)_+)^2. && (26)
\end{aligned}$$

Note that if $h < k$ then $A_{r,k} \subset A_{r,h}$. Take $x \in A_{r,k}$. then $u(x) > k$, and $u(x) - h > k - h$. Therefore

$$\int_{A_{r,k}} ((u-h)_+)^2 \geq \int_{A_{r,k}} (k-h)^2 = (k-h)^2 |A_{r,k}| \tag{27}$$

and

$$|A_{r,k}| \leq \frac{1}{(k-h)^2} \int_{A_{r,k}} ((u-h)_+)^2 \leq \frac{1}{(k-h)^2} \int_{A_{r,h}} ((u-h)_+)^2. \tag{28}$$

for all $h < k$. Plugging this back into 26 we get

$$\begin{aligned}
\int_{A_{r,k}} ((u-k)_+)^2 &\leq \frac{\tilde{c}}{(R-r)^2(k-h)^{4/n}} \left(\int_{A_{r,h}} ((u-h)_+)^2\right)^{2/n} \int_{A_{R,k}} ((u-k)_+)^2 && (29)\\
&\leq \frac{\tilde{c}}{(R-r)^2(k-h)^{4/n}} \left(\int_{A_{R,h}} ((u-h)_+)^2\right)^{2/n} \int_{A_{R,h}} ((u-h)_+)^2 && (30)\\
&\leq \frac{\tilde{c}}{(R-r)^2(k-h)^{4/n}} \left(\int_{A_{R,h}} ((u-h)_+)^2\right)^{(1+2/n)} && (31)
\end{aligned}$$

Next lecture we will actually do the induction argument.