Lecture 22: The mean value inequality for uniformly elliptic operators part II

1 The mean value inequality: Iterative argument continued

In this lecture we will complete the proof of the mean value inequality. Last time we had

$$\int_{A_{r,k}} ((u-k)_+)^2 \leq \frac{\tilde{c}}{(R-r)^2(k-h)^{4/n}} \left(\int_{A_{R,h}} ((u-h)_+)^2 \right)^{(1+2/n)} \tag{1}$$

for all $h < k$ and $r < R$. Define $r_m = 1 + 2^{-m}$ and $k_m = (2 - 2^{-m})k$ for some constant k. Applying the above inequality to $r_m, r_{m+1}, k_m, k_{m+1}$ gives

$$\int_{A_{r_{m+1},k_{m+1}}} ((u-k_{m+1})_+)^2 \leq \frac{\tilde{c}}{(r_m - r_{m+1})^2(k_{m+1}-k_m)^{4/n}} \left(\int_{A_{r_m,k_m}} ((u-k_m)_+)^2 \right)^{(1+2/n)}. \tag{2}$$

Define $\phi(m) = \left(\int_{A_{r_m,k_m}} ((u-k_m)_+)^2 \right)^{1/2}$ and $\epsilon = 2/n$, so

$$\phi(m+1) \leq \frac{\sqrt{\tilde{c}}}{(r_m - r_{m+1})(k_{m+1}-k_m)^{2/n}} (\phi(m))^{1+\epsilon}. \tag{3}$$

Substituting for r_m, k_m and renaming $c = \sqrt{\tilde{c}}$ gives

$$\phi(m+1) \leq \frac{2^{m+1}c}{(2^{-m-1}k)^{2/n}} (\phi(m))^{1+\epsilon}. \tag{4}$$

Now use induction to show that $\phi(n) \to 0$ as $n \to \infty$. Suppose there is some constant $\gamma > 1$ with $\phi(m) \leq \frac{\phi(0)}{\gamma^m}$. Then

$$\begin{aligned} \phi(m+1) &\leq \frac{2^{m+1}c}{(2^{-m-1}k)^{2/n}} \left(\frac{\phi(0)}{\gamma^m} \right)^{1+\epsilon} && (5) \\ &\leq \left(\frac{2^{m+1}c}{(2^{-m-1}k)^{2/n}} \left(\frac{\phi(0)}{\gamma^m} \right)^{\epsilon} \right) \frac{\phi(0)}{\gamma^m} && (6) \end{aligned}$$

so if

$$\left(\frac{2^{m+1} c}{(2^{-m-1} k)^{2/n}} \left(\frac{\phi(0)}{\gamma^m} \right)^{\epsilon} \right) \leq \frac{1}{\gamma} \tag{7}$$

then we get $\phi(n) \leq \frac{\phi(0)}{\gamma^n}$ for all n. It suffices to pick $\gamma > 2^{\frac{1+\epsilon}{\epsilon}}$ and $k > (2^{1+\epsilon} c \gamma (\phi(0))^{\epsilon})^{\frac{1}{\epsilon}} = 2k' \phi(0)$for appropriate k'. Therefore

$$\lim_{n \to \infty} \phi(n) \leq \lim_{n \to \infty} \frac{\phi(0)}{\gamma^n} = 0, \tag{8}$$

so

$$\int_{A_{r_m, k_m}} ((u - k_m)_+)^2 \to 0 \text{ as } n \to \infty. \tag{9}$$

Note that $\lim r_m = 1$, and $\lim k_m = 2k$ so we get

$$\int_{A_{1,2k}} ((u - 2k)_+)^2 = 0 \tag{10}$$

and conclude that $u \leq 2k$ on B_1. Putting in our value for k we obtain

$$\sup_{B_1(x_0)} u \leq (2^{1+\epsilon} c)^{\frac{1}{\epsilon}} \phi(0), \tag{11}$$

and, writing out $\phi(0)$ and ϵ,

$$\sup_{B_1(x_0)} u \leq k' \left(\int_{A_{2,0}} (u_+)^2 \right)^{1/2} \tag{12}$$

$$\leq k' \left(\int_{B_2(x_0)} u^2 \right)^{1/2}. \tag{13}$$

This is the mean value inequality.