Lecture Three: The Hopf Maximum Principle

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In this lecture we will state and prove the Hopf Maximum Principle.

Theorem 1.1 *If u is an harmonic function on the closure of $B_r(0) \subset \mathbf{R}^n$, and x_0 on the boundary of $B_r(0)$ is a strict maximum of u (ie $u(x_0) > u(y)$ for all $y \neq x_0$) then*

$$\frac{\partial u}{\partial \mathbf{n}}(x_0) \geq \frac{k}{r}(u(x_0) - u(0)) \tag{1}$$

for some strictly positive dimensional constant k.

Proof We prove this from the maximum principle. First consider the case $r = 1$. Let $v(x) = e^{-\alpha|x|^2} - e^{-\alpha}$, so $v = 0$ on $\partial B_1(0)$ and $v > 0$ on the interior. Define

$$w : \mathbf{R}^n \longrightarrow \mathbf{R} \text{ by } w(x) = |x|^2$$

and

$$f : \mathbf{R} \longrightarrow \mathbf{R} \text{ by } f(t) = e^{-\alpha t} - e^{-\alpha}$$

so that $v = f(w)$. Now consider

$$\begin{aligned} \triangle f(w) &= f''(w)|\nabla w|^2 + f'(w)\triangle w \\ &= 4\alpha^2 e^{-\alpha|x|^2}|x|^2 - 2n\alpha e^{-\alpha|x|^2}. \end{aligned}$$

Picking $\alpha = 4n$ and restricting v to $1 \geq |x| \geq 1/2$ we obtain

$$\begin{aligned} \triangle v &\geq 2\alpha e^{-\alpha}(\alpha/2 - n) \\ &\geq 8n^2 e^{-4n}. \end{aligned}$$

Now apply this to u. On the annulus $B_1 \setminus B_{1/2}$

$$\triangle(u + \epsilon v) = \epsilon \triangle v > 0, \tag{2}$$

so $u + \epsilon v$ is sub-harmonic on this anulus, and the maximum principle applies. Therefore the maximum of $u + \epsilon v$ on the annulus $B_1 \setminus B_{1/2}$ occurs on the boundary. Recall that u has a strict maximum on the outer boundary, so if we choose ϵ very small we can arrange that $u + \epsilon v$ also takes it's maximum on the outer boundary. For this we need

$$u(x_0) + \epsilon v(x_0) \geq \max_{\partial B_{1/2}}(u(x) + \epsilon v(x))$$

so that

$$u(x_0) \geq \max_{\partial B_{1/2}} u(x) + \epsilon(e^{-n} - e^{-4n}).$$

We can choose

$$\epsilon = \frac{u(x_0) - \max_{\partial B_{1/2}} u(x)}{2(e^{-n} - e^{-4n})}. \tag{3}$$

We know that $u + \epsilon v$ has a maximum on the outer boundary and it has to be at x_0 (since $v = 0$ on the outer boundary). It follows that

$$\frac{\partial(u + \epsilon v)}{\partial n} \geq 0$$

and therefore that

$$\frac{\partial u}{\partial n}(x_0) \geq -\epsilon \frac{\partial v}{\partial n}.$$

Calculating $\frac{\partial v}{\partial n}$ and substituting in for ϵ we obtain

$$\frac{\partial u}{\partial n}(x_0) \geq \frac{8ne^{-4n}}{2(e^{-n} - e^{-4n})}(u(x_0) - \max_{\partial B_{1/2}} u(x)). \tag{4}$$

Finally we apply the Harnack inequality to get this in terms of $u(0)$. Define $w(x)$ by $w(x) = u(x_0) - u(x)$. Note that w is harmonic and non-negative, therefore the Harnack inequality holds, and we get

$$w(0) \leq \max_{B_{1/2}(0)} w(x) \leq C(n) \min_{B_{1/2}(0)} w(x)$$

for an appropriate dimensional constant $C(n)$. Therefore

$$\frac{u(x_0) - u(0)}{C(n)} \leq (u(x_0) - \max_{B_{1/2}(0)} u(x)). \tag{5}$$

Substituting this into (4) we obtain

$$\frac{\partial u}{\partial n}(x_0) \geq \frac{8ne^{-4n}}{2C(n)(e^{-n} - e^{-4n})}(u(x_0) - u(0)). \tag{6}$$

This completes the proof for $r = 1$. We will get the general case by scaling. If u is harmonic on $B_r(0)$ and we define $\tilde{u}(y) = u(ry)$ then $\tilde{u}$ is harmonic on $B_1(0)$. Also if $x_0 \in \partial B_r(0)$ is a strict maximum of u then $\tilde{x}_0 = x_0/r$ is a strict maximum of $\tilde{u}$ on the boundary. Therefore

$$\frac{\partial \tilde{u}}{\partial n}(\tilde{x}_0) \geq \frac{8ne^{-4n}}{2C(n)(e^{-n} - e^{-4n})}(\tilde{u}(\tilde{x}_0) - \tilde{u}(0)) \tag{7}$$

$$\geq \frac{8ne^{-4n}}{2C(n)(e^{-n} - e^{-4n})}(u(x_0) - u(0)). \tag{8}$$

By the chain rule $\frac{\partial \tilde{u}}{\partial n}(\tilde{x}_0) = r\frac{\partial u}{\partial n}(x_0)$, so

$$\frac{\partial u}{\partial n}(x_0) \geq \frac{1}{r}\frac{8ne^{-4n}}{2C(n)(e^{-n} - e^{-4n})}(u(x_0) - u(0)) \tag{9}$$

as required. ∎