Lecture Four: The Poincare Inequalities

In this lecture we introduce two inequalities relating the integral of a function to the integral of it's gradient. They are the Dirichlet-Poincare and the Neumann-Poincare inequalities.

1 The Dirichlet-Poincare Inequality

Theorem 1.1 *If $u : B_r \to \mathbb{R}$ is a C^1 function with $u = 0$ on ∂B_r then*

$$\int_{B_r} u^2 \leq C(n) r^2 \int_{B_r} |\nabla u|^2. \tag{1}$$

We will prove this for the case $n = 1$. Here the statement becomes

$$\int_{-r}^{r} f^2 \leq k r^2 \int_{-r}^{r} (f')^2 \tag{2}$$

where f is a C^1 function satisfying $f(-r) = f(r) = 0$. By the Fundamental Theorem of Calculus

$$f(s) = \int_{-r}^{s} f'(x). \tag{3}$$

Therefore

$$|f(s)| \leq \int_{-r}^{s} |f'(x)|. \tag{4}$$

Recall the Cauchy-Schwarz inequality $\left(\int hg \leq \left(\int h^2\right)^{1/2} \left(\int g^2\right)^{1/2}\right)$. Apply this with $h = 1$, $g = |f'|$ to get

$$|f(s)| \leq \left(\int_{-r}^{s} (f')^2\right)^{1/2} (r+s)^{1/2} \leq \left(\int_{-r}^{r} (f')^2\right)^{1/2} (2r)^{1/2}\ . \tag{5}$$

Squaring both sides gives

$$|f(s)|^2 \leq 2r \int_{-r}^{r} (f'(s))^2, \tag{6}$$

and finally we integrate over $[-r, r]$ to give

$$\int_{-r}^{r} |f(s)|^2 \leq 4r^2 \int_{-r}^{r} |f'(s)|^2 \tag{7}$$

as required.

2 The Nueman-Poincare Inequality

Theorem 2.1 *If u is C^1 on B_r, and we define $A = \frac{1}{volB_r} \int_{B_r} u$ then*

$$\int_{B_r} (u - A)^2 \leq C(n) r^2 \int_{B_r} |\nabla u|^2. \tag{8}$$

Again we give the proof in the case $n = 1$.

Take a differentiable function f with $A = \frac{1}{2r} \int_{-r}^{r} f$. Note by the intermediate value theorem that there is a point c in $[-r, r]$ with $f(c) = A$. We have

$$f(s) = f(c) + \int_c^s f'(s).$$

From this we get $|f(s) - A| \leq \int_c^s |f'(t)| \leq \int_{-r}^{r} |f'(t)|$. Apply Cauchy-Schwarz again to give

$$|f(s) - A| \leq 2r \left(\int_{-r}^{r} (f')^2 \right)^{1/2}. \tag{9}$$

Squaring and integrating then gives our result.

$$\int_{-r}^{r} (f(s) - A)^2 \leq (2r)^2 \int_{-r}^{r} |f'(s)|^2. \tag{10}$$

It is not difficult to extend these proofs to higher dimensional cubes.

There is another interesting fact related to the Neumann-Poincare inequality. If we define $g(x) = \int_{B_r} (u - x)^2$ we can rearrange to get $g(x) = \int_{B_r} u^2 - 2x \int_{B_r} u - x^2$. We then find the minimum by taking the derrivative, and see that it occurs at $x = \frac{\int_{B_r} u}{\mathrm{vol} B_r}$, so the mean really is the best constant approximation to a function.