Lecture Five: The Cacciopolli Inequality

1 The Cacciopolli Inequality

The Cacciopolli (or Reverse Poincare) Inequality bounds similar terms to the Poincare inequalities studied last time, but the other way around. The statement is this.

Theorem 1.1 *Let $u : B_{2r} \to \mathbf{R}$ satisfy $u \triangle u \geq 0$. Then*

$$\int_{B_r} |\nabla u|^2 \leq \frac{4}{r^2} \int_{B_{2r} \setminus B_r} u^2. \tag{1}$$

First prove a Lemma.

Lemma 1.2 *If $u : B_{2r} \to \mathbf{R}$ satisfies $u \triangle u \geq 0$, and $\phi : B_{2r} \to \mathbf{R}$ is non-negative with $\phi = 0$ on ∂B_{2r}, then*

$$\int_{B_{2r}} \phi^2 |\nabla u|^2 \leq 4 \int_{B_{2r}} |u|^2 |\nabla \phi|^2. \tag{2}$$

Proof Consider

$$0 \leq \int_{B_{2r}} \phi^2 u \triangle u. \tag{3}$$

Clearly $\int_{\partial B_{2r}} \phi^2 u \nabla u \cdot dS = 0$, so apply Stokes' theorem to get $\int_{B_{2r}} \phi^2 u \triangle u + \int_{B_{2r}} \nabla(\phi^2 u) \cdot \nabla u = 0$. From this

$$0 \leq - \int_{B_{2r}} \nabla(\phi^2 u) \nabla u = -2 \int_{B_{2r}} \phi u \nabla \phi \cdot \nabla u - \int_{B_{2r}} \phi^2 |\nabla u|^2, \tag{4}$$

and so

$$\int_{B_{2r}} \phi^2 |\nabla u|^2 \leq -2 \int_{B_{2r}} \phi u \nabla \phi \nabla u \tag{5}$$

$$\leq 2 \int_{B_{2r}} \phi |u| |\nabla \phi| |\nabla u|. \tag{6}$$

Recall the inequality $\int fg \leq \left(\int f^2\right)^{1/2}\left(\int g^2\right)^{1/2}$ for any functions f and g (this is one form of the Cauchy-Schwarz inequality), and apply it above to get

$$\int_{B_{2r}} \phi^2|\nabla u|^2 \leq 2\left(\int_{B_{2r}} \phi^2|\nabla u|^2\right)^{1/2}\left(\int_{B_{2r}} |u|^2|\nabla\phi|^2\right)^{1/2}. \tag{7}$$

Dividing and squaring then gives

$$\int_{B_{2r}} \phi^2|\nabla u|^2 \leq 4\int_{B_{2r}} |u|^2|\nabla\phi|^2. \quad \blacksquare \tag{8}$$

To complete the proof of theorem 1.1 pick

$$\phi(x) = \begin{cases} 1 & \text{if } |x| \leq r; \\ \frac{2r-|x|}{r} & \text{if } r < x \leq 2r, \end{cases}$$

so $|\nabla\phi| = 0$ on B_r and $|\nabla\phi| = 1/r$ on $B_{2r} \setminus B_r$. Substitute this into the lemma to obtain the result, namely

$$\int_{B_r} |\nabla u|^2 \leq \frac{4}{r^2}\int_{B_{2r}\setminus B_r} u^2. \tag{9}$$

2 Applications of the Cacciopolli Inequality

2.1 Bounding the growth of a harmonic function

One nice consequence of the Cacciopolli Inequality is the following inequality bounding the rate at which a harmonic function can decay.

Proposition 2.1 *There are strictly positive dimensional constants $k(n)$ such that*

$$\int_{B_{2r}} u^2 \geq (1+k(n))\int_{B_r} u^2 \tag{10}$$

for all harmonic functions $u : B_{2r} \to \mathbb{R}$.

Proof Let ϕ be a test function as before, and consider

$$\begin{aligned} \int_{B_{2r}} |\nabla(\phi u)|^2 &= \int_{B_{2r}} |\phi\nabla u + u\nabla\phi|^2 \\ &= \int_{B_{2r}} \phi^2|\nabla u|^2 + u^2|\nabla\phi|^2 + 2u\phi\nabla\phi\cdot\nabla u. \end{aligned}$$

Apply Cauchy-Schwarz and lemma 1.2 to get

$$\begin{aligned}\int_{B_{2r}} |\nabla(\phi u)|^2 &\leq \int_{B_{2r}} \phi^2|\nabla u|^2 + \int_{B_{2r}} u^2|\nabla\phi|^2 + 2\left(\int_{B_{2r}} \phi^2|\nabla u|^2\right)^{1/2}\left(\int_{B_{2r}} u^2|\nabla\phi|^2\right)^{1/2} \\ &\leq 2\int_{B_{2r}} \phi^2|\nabla u|^2 + 2\int_{B_{2r}} u^2|\nabla\phi|^2. \\ &\leq 10\int_{B_{2r}} u^2|\nabla\phi|^2.\end{aligned}$$

Now make the same choice of ϕ as before to give

$$\int_{B_{2r}} |\nabla(\phi u)|^2 \leq \frac{10}{r^2}\int_{B_{2r}\setminus B_r} u^2 \tag{11}$$

and apply Dirichlet-Poincare to the left hand side to get

$$\frac{1}{C(n)r^2}\int_{B_{2r}} \phi^2 u^2 \leq \frac{10}{r^2}\int_{B_{2r}\setminus B_r} u^2. \tag{12}$$

Since $(\phi u)^2$ is a positive function we can reduce the area of the integration, therefore

$$k(n)\int_{B_r} \phi^2 u^2 \leq \int_{B_{2r}\setminus B_r} u^2. \tag{13}$$

for $k(n) = \frac{1}{10C(n)}$. Finally note that $\phi = 1$ on B_r, so

$$k(n)\int_{B_r} u^2 \leq \int_{B_{2r}\setminus B_r} u^2, \tag{14}$$

and

$$(1 + k(n))\int_{B_r} u^2 \leq \int_{B_{2r}} u^2. \tag{15}$$

This completes the proof. ∎

2.2 Bounding the growth of the energy of a harmonic function

We will now prove a similar inequality for the Dirichlet energy of a harmonic function.

Proposition 2.2 *There are dimensional constants $c(n)$ such that*

$$\int_{B_{2r}} |\nabla u|^2 \geq (1 + \theta(n))\int_{B_r} |\nabla u|^2. \tag{16}$$

for all harmonic functions $u : B_{2r} \to \mathbb{R}$. .

Proof It suffices to show that

$$c(n)\int_{B_r}|\nabla u|^2 \leq \int_{B_{2r}\backslash B_r}|\nabla u|^2. \tag{17}$$

To do this we use two inequalities. Firstly we will state and use without proof the Neumann-Poincare inequality for an annulus, namely if $A = \frac{1}{\mathrm{vol}_{B_{2r}\backslash B_r}}\int_{B_{2r}\backslash B_r} u$ then

$$\int_{B_{2r}\backslash B_r}(u-A)^2 \leq d(n)r^2\int_{B_{2r}\backslash B_r}|\nabla u|^2. \tag{18}$$

Secondly we use Cacciopolli, noting that if $\triangle u = 0$ then $\triangle(u+A) = 0$, and $\nabla(u+A) = \nabla u$, to give

$$r^2\int_{B_r}|\nabla u|^2 \leq 4\int_{B_{2r}\backslash B_r}(u-A)^2. \tag{19}$$

Together (15) and (16) give

$$\frac{1}{4d(n)}\int_{B_r}|\nabla u|^2 \leq \int_{B_{2r}\backslash B_r}|\nabla u|^2 \tag{20}$$

as required. ∎