Lecture Eight: Maximum principles and gradient estimates

1 The Maximum Principle for more general operators

Let $u : B_r \to \mathbb{R}^n$ be a C^2 function, and let L be a uniformly elliptic differential operator taking

$$Lu = \sum_{i,j} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} \tag{1}$$

for some real $n \times n$ symmetric matrix A with continuously differentiable entries. If x is an internal maximum of u then

$$\nabla u = 0 \text{ and } Lu \leq 0$$

at x. When A is the identity matrix this is exactly the maximum principle from lecture 1. If not then we pick coordnates at x so that $A(x)$ is diagonal. Since all the eigenvalues of $A(x)$ are positive we have

$$Lu = \sum_i b_i \frac{\partial^2 u}{\partial x_i^2}|_{(x)} \tag{2}$$

for some collection of positive constants b_i. Since x is a maximum $\frac{\partial^2 u}{\partial x_i^2}|_{(x)}$ is non-positive for all i, and $Lu \leq 0$ as expected.

2 The gradient estimate for L-harmonic functions

Recall the gradient estimate

$$\sup_{B_r} |\nabla u| \leq \frac{c(n)}{r} \sup_{B_{2r}} |u|$$

for harmonic functions u. We will prove a similar estimate for uniformly elliptic operators.

Proposition 2.1 *Let L be a uniformly elliptic operator as above, with*

$$\lambda|\mathbf{v}|^2 \leq \mathbf{v}\cdot(A\mathbf{v}) \leq \Lambda|\mathbf{v}|^2$$

for some real $0 < \lambda \leq \Lambda$. There are constants C which depend only on the operator and the dimension of the space such that

$$\sup_{B_r} |\nabla u| \leq \frac{C}{r} \sup_{B_{2r}} |u| \tag{3}$$

for all L-harmonic functions u on B_{2r}.

Proof This proof follows essentially the same steps as the proof for the earlier gradient estimate from lecture 2, but each step is now more complicated. When constants are intrioduced it is important to noctive that they depend only on A, n, λ and Λ. As before we will first prove the case $r = 1$, and then extend to general r.

Step 1. One key part of the proof in the harmonic case was the Bochner formula. We will prove a similar result for L harmonic functions. Calculate

$$\frac{\partial^2}{\partial x_i \partial x_j}|\nabla u|^2 = \frac{\partial}{\partial x_i}\sum_k 2\frac{\partial^2 u}{\partial x_j \partial x_k}\frac{\partial u}{\partial x_k} \tag{4}$$

$$= 2\sum_k \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}\frac{\partial u}{\partial x_k} + \frac{\partial^2 u}{\partial x_j \partial x_k}\frac{\partial^2 u}{\partial x_i \partial x_k}. \tag{5}$$

Therefore

$$L|\nabla u|^2 = 2\sum_{ijk} A_{ij}\frac{\partial^3 u}{\partial_i \partial x_j \partial x_k}\frac{\partial u}{\partial x_k} + 2\sum_k \left(\sum_{ij} A_{ij}\frac{\partial^2 u}{\partial x_j \partial x_k}\frac{\partial^2 u}{\partial x_i \partial x_k}\right). \tag{6}$$

Let $v_k = \frac{\partial u}{\partial x_k}$. The last term is $\nabla v_k \cdot A\nabla v_k \geq \lambda|\nabla v_k|^2$ by uniform ellipticity. Substituting in gives

$$L|\nabla u|^2 \geq 2\sum_{ijk} A_{ij}\frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}\frac{\partial u}{\partial x_k} + 2\lambda\sum_{ik}\left(\frac{\partial^2 u}{\partial x_i \partial x_k}\right)^2. \tag{7}$$

Now work on the first term.

$$\sum_{ijk} A_{ij}\frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}\frac{\partial u}{\partial x_k} = \sum_{ijk}\frac{\partial}{\partial x_k}\left(A_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}\right)\frac{\partial u}{\partial x_k} - \sum_{ijk}\frac{\partial A_{ij}}{\partial x_k}\frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial u}{\partial x_k}$$

$$= \sum_k \frac{\partial}{\partial x_k}(Lu)\frac{\partial u}{\partial x_k} - \sum_{ijk}\frac{\partial A_{ij}}{\partial x_k}\frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial u}{\partial x_k}$$

$$= -\sum_{ijk}\frac{\partial A_{ij}}{\partial x_k}\frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial u}{\partial x_k}$$

since $Lu = 0$. Together with (7) this gives

$$L|\nabla u|^2 \geq -2 \sum_{ijk} \frac{\partial A_{ij}}{\partial x_k} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_k} + 2\lambda \sum_{ik} \left(\frac{\partial^2 u}{\partial x_i \partial x_k} \right)^2 . \tag{8}$$

Let $c_{ij} = \sum_k \frac{\partial A_{ij}}{\partial x_k} \frac{\partial u}{\partial x_k}$ and $d_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$. We can re-write (8) as

$$L|\nabla u|^2 \geq -2 \sum_{ij} c_{ij} d_{ij} + 2\lambda \sum_{ij} d_{ij}^2. \tag{9}$$

Note that

$$\lambda d_{ij}^2 - 2c_{ij}d_{ij} + \frac{c_{ij}^2}{\lambda} = \left(\frac{c_{ij}}{\sqrt{\lambda}} - \sqrt{\lambda} d_{ij} \right)^2 \geq 0, \tag{10}$$

and so

$$\lambda d_{ij}^2 - 2c_{ij}d_{ij} \geq -\frac{1}{\lambda} c_{ij}^2. \tag{11}$$

Apply this to (9) to get

$$L|\nabla u|^2 \geq -\frac{1}{\lambda} \sum_{ij} c_{ij}^2 + \lambda \sum_{i,j} d_{ij}^2, \tag{12}$$

or, in the old notation,

$$\begin{aligned} L|\nabla u|^2 &\geq -\frac{1}{\lambda} \sum_{ij} \left(\sum_k \frac{\partial A_{ij}}{\partial x_k} \frac{\partial u}{\partial x_k} \right)^2 + \lambda \sum_{ik} \left(\frac{\partial^2 u}{\partial x_i \partial x_k} \right)^2 \\ &\geq -\frac{1}{\lambda} \sum_{ij} (\nabla A_{ij} \cdot \nabla u)^2 + \lambda \sum_{ik} \left(\frac{\partial^2 u}{\partial x_i \partial x_k} \right)^2 \\ &\geq -\frac{1}{\lambda} \sum_{ij} |\nabla A_{ij}|^2 |\nabla u|^2 + \lambda \sum_{ik} \left(\frac{\partial^2 u}{\partial x_i \partial x_k} \right)^2 \\ &\geq -\frac{1}{\lambda} \left(\sum_{ij} |\nabla A_{ij}|^2 \right) |\nabla u|^2 + \lambda \sum_{ik} \left(\frac{\partial^2 u}{\partial x_i \partial x_k} \right)^2 . \end{aligned}$$

Pick $c_1 = \sup_{B_{2r}} \frac{1}{\lambda} \left(\sum_{ij} |\nabla A_{ij}|^2 \right) > 0$. We have

$$L|\nabla u|^2 \geq -c_1 |\nabla u|^2 + \lambda \sum_{ik} \left(\frac{\partial^2 u}{\partial x_i \partial x_k} \right)^2 . \tag{13}$$

Step 2. We will also need to estimate

$$
\begin{aligned}
L(u^2) &= \sum_{i,j} A_{ij} \frac{\partial u^2}{\partial x_i \partial x_j} \\
&= A_{ij} \frac{\partial}{\partial x_i}\left(2u \frac{\partial u}{\partial x_j}\right) \\
&= 2uA_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + 2A_{ij} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} \\
&= 2A_{ij} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i}
\end{aligned}
$$

since $Lu = 0$. Now apply uniform ellipticity to get

$$
L(u^2) \geq 2\lambda |\nabla u|^2. \tag{14}
$$

Step 3. Assume $r = 1$, and pick ϕ a test function with $\phi = 0$ on ∂B_2, and $\phi > 0$ on the interior. We need a bound for $L(\phi^2 |\nabla u|^2)$. Calculate

$$
\begin{aligned}
L(\phi^2|\nabla u|^2) &= A_{ij} \frac{\partial^2 (\phi^2 |\nabla u|^2)}{\partial x_i \partial x_j} && (15) \\
&= \phi^2 A_{ij} \frac{\partial^2 |\nabla u|^2}{\partial x_i \partial x_j} + |\nabla u|^2 A_{ij} \frac{\partial^2 \phi^2}{\partial x_i \partial x_j} + 2A_{ij}\phi \frac{\partial \phi}{\partial x_i} \frac{\partial |\nabla u|^2}{\partial x_j} && (16) \\
&= \phi^2 L(|\nabla u|^2) + L(\phi^2)|\nabla u|^2 + 4A_{ij}\phi \frac{\partial u}{\partial x_k} \frac{\partial \phi}{\partial x_i} \frac{\partial^2 u}{\partial x_j \partial x_k}. && (17)
\end{aligned}
$$

Note that $L\phi^2$ is bounded on B_2, say $L\phi^2 \geq c_2$. Applying this bound together with inequality (13) we get

$$
L(\phi^2|\nabla u|^2) \geq (c_2 - c_1\phi^2)|\nabla u|^2 + \lambda \phi^2 \left(\frac{\partial^2 u}{\partial x_j \partial x_k}\right)^2 + 4A_{ij}\phi \frac{\partial u}{\partial x_k} \frac{\partial \phi}{\partial x_i} \frac{\partial^2 u}{\partial x_j \partial x_k}. \tag{18}
$$

The function ϕ^2 is also bounded on B_2, so we can pick a positive constant c_3 such that $c_1 - k\phi^2 \geq -c_3$. Then

$$
L(\phi^2|\nabla u|^2) \geq -c_3|\nabla u|^2 + \lambda \phi^2 \left(\frac{\partial^2 u}{\partial x_j \partial x_k}\right)^2 + 4A_{ij}\phi \frac{\partial u}{\partial x_k} \frac{\partial \phi}{\partial x_i} \frac{\partial^2 u}{\partial x_j \partial x_k}. \tag{19}
$$

Now set $\gamma_{jk} = \phi \frac{\partial^2 u}{\partial x_j \partial x_k}$ and $\delta_{jk} = A_{ij} \frac{\partial u}{\partial x_k} \frac{\partial \phi}{\partial x_i}$, and rewrite as

$$
L(\phi^2|\nabla u|^2) \geq -c_3|\nabla u|^2 + \lambda \gamma_{ik}^2 + 4\gamma_{jk}\delta_{jk}. \tag{20}
$$

Use an absorbing inequality to show that

$$L(\phi^2|\nabla u|^2) \geq -c_3|\nabla u|^2 - c_4\delta_{jk}^2 \tag{21}$$

for some positive constant c_4. Substitute in for δ_{jk} and we have

$$\begin{aligned} L(\phi^2|\nabla u|^2) &\geq -c_3|\nabla u|^2 - c_4\sum_{j.k}\left(\sum_i A_{ij}\frac{\partial u}{\partial x_k}\frac{\partial \phi}{\partial x_i}\right)^2 \\ &\geq -c_3|\nabla u|^2 - c_4 n\sum_{j.k}\sum_i\left(A_{ij}\frac{\partial u}{\partial x_k}\frac{\partial \phi}{\partial x_i}\right)^2 \end{aligned}$$

since for any collection of real numbers the average of the squares is greater than the square of the averages. The functions A_{ij} are bounded on B_2, so there is a constant k with $|A_{ij}| \leq k$. Thus

$$\begin{aligned} L(\phi^2|\nabla u|^2) &\geq -c_3|\nabla u|^2 - c_4 nk^2\sum_{j.k}\sum_i\left(\frac{\partial u}{\partial x_k}\frac{\partial \phi}{\partial x_i}\right)^2 \\ &\geq -c_3|\nabla u|^2 - c_4 n^2k^2|\nabla\phi|^2|\nabla u|^2, \end{aligned}$$

and since $|\nabla\phi|^2$ is bounded on B_2

$$L(\phi^2|\nabla u|^2) \geq -c_5|\nabla u|^2 \tag{22}$$

for some constant c_5 which depends only on the dimension and L.

Step 4. Apply the maximum principle. By steps 2 and 3

$$L(\phi^2|\nabla u|^2 + \frac{c_5}{2\lambda}u^2) \geq 0, \tag{23}$$

so, by the maximum principle, $\phi^2|\nabla u|^2 + \frac{c_5}{2\lambda}u^2$ has no interior maxima. Therefore it takes it's maximum on the boundary. We have shown that

$$\sup_{B_2}(\phi^2|\nabla u|^2 + \frac{c_5}{2\lambda}u^2) = \sup_{\partial B_2}(\phi^2|\nabla u|^2 + \frac{c_5}{2\lambda}u^2). \tag{24}$$

Recall that ϕ is zero on the boundary, and choose it to be identically one on B_1. Thus

$$\begin{aligned} \sup_{B_1}|\nabla u|^2 &\leq \sup_{B_1}(\phi^2|\nabla u|^2 + \frac{c_5}{2\lambda}u^2) \\ &\leq \sup_{B_2}(\phi^2|\nabla u|^2 + \frac{c_5}{2\lambda}u^2) \\ &\leq \sup_{\partial B_2}(\phi^2|\nabla u|^2 + \frac{c_5}{2\lambda}u^2) \end{aligned}$$

$$\begin{aligned} &\leq \frac{c_5}{2\lambda} \sup_{\partial B_2} u^2 \\ &\leq \frac{c_5}{2\lambda} \sup_{B_2} u^2. \end{aligned}$$

Finally take square roots to obtain

$$\sup_{B_1} |\nabla u| \leq \frac{c_5}{2\lambda} \sup_{B_2} |u| \tag{25}$$

as expected.

step 5. Scale for general r. Assume u is L-harmonic on B_{2r}, and define $u'(\mathbf{x}) = u(r\mathbf{x})$. Then u' is L harmonic on B_2. Plugging u' into (25) we get

$$r \sup_{B_r} |\nabla u| \leq \frac{c_5}{2\lambda} \sup_{B_{2r}} |u| \tag{26}$$

as required. ∎