Lecture Nine: Hopf and Harnack Revisited

1 The Hopf maximum principle for uniformly elliptic operators

The next result that we will generalise is the Hopf Maximum principle. As before we will consider uniformly elliptic operators L taking

$$Lf = A_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}$$

with $\lambda|\mathbf{v}|^2 < \mathbf{v}\cdot A\mathbf{v} \leq \Lambda|\mathbf{v}|^2$ for some real $0 < \lambda \leq \Lambda$.

Theorem 1.1 *(The Hopf Maximum principle for uniformly elliptic operators) Let u be an L harmonic function on $B_r(0)$ with a strict maximum at $x \in \partial B_r(0)$. There are constants C which depend only on L and the dimension such that*

$$\frac{\partial u}{\partial \mathbf{n}}|_{(x)} \geq \frac{C}{r}(u(x) - u(0)). \tag{1}$$

We will actually prove that

$$\frac{\partial u}{\partial \mathbf{n}}|_{(x)} \geq \frac{C}{r}(u(x) - \sup_{B_r} u). \tag{2}$$

Theorem 1.1 will then follow easily once we have a harnack inequaltity for elliptic operators.

Proof This proof is similar to the earlier version, though a bit more complicated. We will prove the case $r = 1$ and claim that the general result follows by scaling exactly as it did for the previous Hopf maximum principle. Let α be a constant, and define

$$v(\mathbf{x}) = e^{-\alpha|\mathbf{x}|^2} - e^{-\alpha}. \tag{3}$$

Calculate

$$\begin{aligned}
Lv &= A_{ij}\frac{\partial^2 v}{\partial x_i \partial x_j} \\
&= A_{ij}\frac{\partial}{\partial x_j}\left(-2\alpha x_i e^{-\alpha|\mathbf{x}|^2}\right) \\
&= \left(-2A_{ii}\alpha + 4A_{ij}\alpha^2 x_i x_j\right)e^{-\alpha|\mathbf{x}|^2} \\
&\geq \left(-2A_{ii}\alpha + 4\alpha^2\lambda|\mathbf{x}|^2\right)e^{-\alpha|\mathbf{x}|^2}
\end{aligned}$$

by uniform elipticity. Restricting to $B_1 \setminus B_{1/2}$ we have

$$Lv \geq \left(\alpha^2 \lambda - 2\alpha A_{ii}\right) e^{-\alpha|\mathbf{x}|^2}, \tag{4}$$

and so picking α large we can get $Lv \geq 0$ on the annulus.

Consider $u + \epsilon v$. Clearly this is subharmonic on $B_1 \setminus B_{1/2}$, so it takes it's maximum on either the inner or the outer boundary. We'll pick ϵ so that it occurs at x. We need

$$u(x) + \epsilon v(x) \geq \sup_{B_r}(u + \epsilon v)$$

Evaluating v gives

$$u(x) \geq \sup_{B_r}(u + \epsilon(e^{-\alpha/4} - e^{-\alpha})),$$

therefore choose

$$\epsilon = \frac{u(x) - \sup_{B_r} u}{2(e^{\alpha/4} - e^{-\alpha})}. \tag{5}$$

Also note that v is zero on the outer boundary, so the maximum of $u + \epsilon v$ is at x. It follows that

$$\left.\frac{\partial(u + \epsilon v)}{\partial \mathbf{n}}\right|_{(x)} \geq 0. \tag{6}$$

Calculate $\left.\frac{\partial v}{\partial \mathbf{n}}\right|_{(x)} = -2\alpha e^{-\alpha}$, substitute in, and rearrange to get

$$\left.\frac{\partial u}{\partial \mathbf{n}}\right|_{(x)} \geq -\epsilon \left.\frac{\partial v}{\partial \mathbf{n}}\right|_{(x)} \tag{7}$$

$$\geq \frac{2\alpha e^{-\alpha}}{2(e^{\alpha/4} - e^{-\alpha})}(u(x) - \sup_{B_r} u). \tag{8}$$

The result then follows from the Harnack inequality. ∎

2 Another proof of the Harnack inequality

We will now give an alternative proof of the Harnack inequality. It is based on a gradient estimate that is slightly stronger than the one we proved.

Proposition 2.1 *Let u be a positive harmonic function on B_{2r}. Then*

$$\sup_{B_r} \frac{|\nabla u|}{u} \leq \frac{c}{r} \tag{9}$$

for some dimensional constant c (ie c depends on the dimension of the space, but not on u).

We will prove this next time.

We can derive the Harnack inequality from this as follows. Pick x and y in B_r, and let γ_1 be the straight path from x to 0, and γ_2 the straight path from 0 to y. Note that $\frac{|\nabla u|}{u} = |\nabla(\log u)|$, and calculate

$$\begin{aligned} |\log u(y) - \log u(x)| &= \left| \int_{\gamma_1} \nabla(\log u) \cdot dx + \int_{\gamma_2} \nabla(\log u) \cdot dx \right| && (10) \\ &\leq |x| \int_0^1 |\nabla(\log u(sx)| ds + |y| \int_0^1 |\nabla(\log u(sy))| ds && (11) \\ &\leq (|x| + |y|) \frac{c}{r} && (12) \\ &\leq 2c. && (13) \end{aligned}$$

Taking exponents

$$e^{-2c} \leq \frac{u(y)}{u(x)} \leq e^{2c}, \tag{14}$$

and so

$$\sup_{B_r} u \leq e^{2c} \inf_{B_r} u \tag{15}$$

as required.