MATH 18.152 - OPTIONAL BONUS PROBLEM

18.152 Introduction to PDEs, Fall 2011	Professor: Jared Speck
Optional Bonus Problem, **Due: at the start of class on 11-29-11**	

I. Consider the *Morawetz vectorfield* $\overline{K}^\mu$ on R^{1+3} defined by

$$\overline{K}^0 = 1 + t^2 + (x^1)^2 + (x^2)^2 + (x^3)^2, \tag{0.0.1}$$

$$\overline{K}^j = 2tx^j, \qquad (j = 1, 2, 3). \tag{0.0.2}$$

a) Show that $\overline{K}$ is future-directed and timelike. Above, (t, x^1, x^2, x^3) are the standard coordinates on R^{1+3}.

b) Show that

$$\partial_\mu \overline{K}_\nu + \partial_\nu \overline{K}_\mu = 4t m_{\mu\nu}, \qquad (\mu, \nu = 0, 1, 2, 3), \tag{0.0.3}$$

where $m_{\mu\nu}$ denotes the Minkowski metric.

Remark 0.0.1. $\overline{K}$ is said to be a *conformal Killing field* of the Minkowski metric because the right-hand side of (0.0.3) is proportional to $m_{\mu\nu}$.

c) Show that

$$m_{\mu\nu} T^{\mu\nu} = -(m^{-1})^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi, \tag{0.0.4}$$

where $T_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m_{\mu\nu} (m^{-1})^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi$ is the energy-momentum tensor corresponding to the linear wave equation, and $T^{\mu\nu} \stackrel{\text{def}}{=} (m^{-1})^{\mu\alpha} (m^{-1})^{\nu\beta} T_{\alpha\beta}$ is the energy-momentum tensor with its indices raised.

d) Show that $\partial_\mu {}^{(\overline{K})}J^\mu = 2t (m^{-1})^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi$ whenever ϕ is a C^2 solution to the linear wave equation $(m^{-1})^{\mu\nu} \partial_\mu \partial_\nu \phi = 0$, where

$${}^{(\overline{K})}J^\mu \stackrel{\text{def}}{=} -T^{\mu\nu} \overline{K}_\nu. \tag{0.0.5}$$

e) Show that $\partial_\mu \widetilde{J}^\mu = 0$ whenever ϕ is a C^2 solution to the linear wave equation $(m^{-1})^{\mu\nu} \partial_\mu \partial_\nu \phi = 0$, where

$$\widetilde{J}^\mu \stackrel{\text{def}}{=} {}^{(\overline{K})}J^\mu - 2t\phi (m^{-1})^{\mu\alpha} \partial_\alpha \phi + \phi^2 (m^{-1})^{\mu\alpha} \partial_\alpha t. \tag{0.0.6}$$

f) Show that

$${}^{(\overline{K})}J^0 = \frac{1}{4} \Big\{ \big[1 + (t+r)^2\big] (\nabla_L \phi)^2 + \big[1 + (t-r)^2\big] (\nabla_{\underline{L}} \phi)^2 + 2\big[1 + t^2 + r^2\big] \not{m}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \Big\}. \tag{0.0.7}$$

Above, $(m^{-1})^{\mu\nu} = -\frac{1}{2}L^\mu \underline{L}^\nu - \frac{1}{2}\underline{L}^\mu L^\nu + \not{m}^{\mu\nu}$ is the standard null decomposition of $(m^{-1})^{\mu\nu}$ from class. In particular, $L^\mu = (1, \frac{x^1}{r}, \frac{x^2}{r}, \frac{x^3}{r})$, $\underline{L}^\mu = (1, -\frac{x^1}{r}, -\frac{x^2}{r}, -\frac{x^3}{r})$, $\nabla_L \phi = \partial_t \phi + \partial_r \phi$, $\nabla_{\underline{L}} \phi = \partial_t \phi - \partial_r \phi$, and $\not{m}^{\mu\nu}\partial_\mu \phi \partial_\nu \phi$ is the square of the Euclidean norm of the *angular* derivatives of ϕ. Here, $r \stackrel{\text{def}}{=} \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$ denotes the standard spherical coordinate on $\mathbb{R}^3$, and ∂_r denotes the standard radial derivative.
Hint: The following expansions in terms of L and $\underline{L}$ may be very helpful:

(0.0.8)
$$\overline{K}^\mu = \frac{1}{2}\Big\{[1 + (r+t)^2]L^\mu + [1 + (r-t)^2]\underline{L}^\mu\Big\},$$
(0.0.9)
$$(1, 0, 0, 0) = \frac{1}{2}(L^\mu + \underline{L}^\nu),$$
(0.0.10)
$$\begin{aligned} {}^{(\overline{K})}J^0 &= T\big(\overline{K}, \frac{1}{2}(L + \underline{L})\big) \\ &= \frac{1}{4}\Big\{[1 + (r+t)^2]T\big(L, L\big) + [1 + (r-t)^2]T\big(\underline{L}, \underline{L}\big) + \big([1 + (r+t)^2] + [1 + (r-t)^2]\big)T\big(L, \underline{L}\big)\Big\}. \end{aligned}$$

f)
Show that

(0.0.11)
$$\widetilde{J}^0 = \frac{1}{4}\Big\{\big[1 + (t+r)^2\big](\nabla_L \phi)^2 + \big[1 + (t-r)^2\big](\nabla_{\underline{L}} \phi)^2 + 2\big[1 + t^2 + r^2\big] \not{m}^{\mu\nu}\partial_\mu \phi \partial_\nu \phi\Big\} + 2t\phi\partial_t \phi - \phi^2.$$

g) Show that

$$-\int_{\mathbb{R}^3} \phi^2 \, d^3x = \frac{2}{3}\int_{\mathbb{R}^3} r\phi\partial_r \phi \, d^3x \tag{0.0.12}$$

whenever ϕ is a C^1, compactly supported function.
Hint: Use the identity $1 = \partial_r r$ together with integration by parts in spherical coordinates and the fact that $d^3x = r^2 \sin\theta \, dr d\theta d\phi$ in spherical coordinates.

h) Use parts **f)** and **g)** to show that

(0.0.13)
$$\widetilde{J}^0 = \frac{1}{4}\Big\{(\nabla_L \phi)^2 + \Big(\nabla_L\big[(t+r)\phi\big]\Big)^2 + (\nabla_{\underline{L}} \phi)^2 + \Big(\nabla_{\underline{L}}\big[(t-r)\phi\big]\Big)^2 + 2\big[1 + t^2 + r^2\big] \not{m}^{\mu\nu}\partial_\mu \phi \partial_\nu \phi\Big\}.$$

i) Finally, with the help of the vectorfield $\widetilde{J}^\mu$, part **e)**, and part **h)**, apply the divergence theorem on an appropriately chosen spacetime region to derive the following conservation law for smooth solutions to the linear wave equation $(m^{-1})^{\mu\nu}\partial_\mu \partial_\nu \phi = 0$:

(0.0.14)

$$\frac{1}{4}\int_{\mathbb{R}^3}\Big\{(\nabla_L\phi)^2+\Big(\nabla_L\big[(t+r)\phi\big]\Big)^2+(\nabla_{\underline{L}}\phi)^2+\Big(\nabla_{\underline{L}}\big[(t-r)\phi\big]\Big)^2+2\big[1+t^2+r^2\big]\ \not{m}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Big\}\,d^3x$$
$$=\frac{1}{4}\int_{\mathbb{R}^3}\Big\{(\nabla_L\phi)^2+\Big(\nabla_L\big[(t+r)\phi\big]\Big)^2+(\nabla_{\underline{L}}\phi)^2+\Big(\nabla_{\underline{L}}\big[(t-r)\phi\big]\Big)^2+2\big[1+t^2+r^2\big]\ \not{m}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Big\}\,d^3x\Big|_{t=0},$$

where the left-hand side is evaluated at time t, and right-hand side is evaluated at time $t=0$. For simplicity, at each fixed t, you may assume that there exists an $R>0$ such that $\phi(t,x)$ vanishes whenever $|x|\geq R$.

Remark 0.0.2. Note that the right-hand side of (0.0.14) can be computed in terms of the initial data alone. Note also that the different *null derivatives* of ϕ appearing on the left-hand side of (0.0.14) carry different weights. In particular, $\nabla_L\phi$ and the angular derivatives of ϕ have larger weights than $\nabla_{\underline{L}}\phi$. These larger weights are strongly connected to the following fact, whose full proof requires additional methods going beyond this course: $\nabla_L\phi$ and the angular derivatives of ϕ decay faster in t compared to $\nabla_{\underline{L}}\phi$.

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