MATH 18.152 - PROBLEM SET # 1

18.152 Introduction to PDEs, Fall 2011	Professor: Jared Speck

Problem Set # 1, **Due: at the start of class on 9-15-11**

I. Let $\Omega \subset \mathbb{R}^n$ be domain with a smooth boundary $\partial\Omega$. Let $u, v \in C^2(\overline{\Omega})$, where $\overline{\Omega}$ denotes the closure of Ω. Show that the *Green identity* holds:

$$\int_{\Omega} u(x)\Delta v(x) - v(x)\Delta u(x)\, d^n x = \int_{\partial\Omega} u(\sigma)\nabla_{\hat{\mathbf{N}}(\sigma)} v(\sigma) - v(\sigma)\nabla_{\hat{\mathbf{N}}(\sigma)} u(\sigma)\, d\sigma, \tag{0.0.1}$$

where $\hat{\mathbf{N}}(\sigma)$ is the outward unit-normal to $\partial\Omega$ at σ, and $\Delta \overset{\text{def}}{=} \sum_{i=1}^n \partial_i^2$ is the Laplace operator in $\mathbb{R}^n$.

Hint: Apply the divergence theorem to the vectorfield $\mathbf{F} \overset{\text{def}}{=} u\nabla v - v\nabla u$, and use the fact that $\Delta = \nabla \cdot \nabla \overset{\text{def}}{=} \text{div} \circ \text{grad}$.

II. Prove that if ϵ is a number satisfying $0 < \epsilon < 1/2$, then $f(x) \overset{\text{def}}{=} \sin(x)\frac{\ln(x^2+1)}{|x|^{1-\epsilon}}$ satisfies $f \in L^2(\mathbb{R})$, that is, that $\int_{\mathbb{R}} |f(x)|^2\, dx < \infty$.

Hint: Do not try to precisely evaluate the integral! There are three bad spots to worry about: $x = 0$, and $x = \pm\infty$. Try looking in the "improper integrals" section of your old calculus book if you get stuck.

III. Let V be a vector space over $\mathbb{R}$ (think of $V \simeq \mathbb{R}^n$ if you are unfamiliar with the abstract notion of a vector space). Let $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$, $(v, w) \to \langle v, w \rangle$ be a "bilinear function" with the following properties:

- $\langle v, w \rangle = \langle w, v \rangle$
- $\langle v, v \rangle > 0$ unless $v = 0$, in which case $\langle 0, 0 \rangle = 0$.
- If $a, b \in \mathbb{R}$ and $v, \widetilde{v}, w \in V$, then $\langle av + b\widetilde{v}, w \rangle = a\langle v, w \rangle + b\langle \widetilde{v}, w \rangle$.

The above function $\langle , \rangle$ is an abstract version of the "dot-product" from vector calculus. Also, define the norm of a vector v by

$$\|v\| \overset{\text{def}}{=} |\langle v, v \rangle|^{1/2}. \tag{0.0.2}$$

The quantity (0.0.2) is a measure of the size of v.

Show that the Cauchy-Schwartz inequality holds for all vectors v, w :

$$|\langle v, w \rangle| \leq \|v\|\|w\|. \tag{0.0.3}$$

Then use (0.0.3) to prove the triangle inequality:

$$\|v + w\| \leq \|v\| + \|w\|. \tag{0.0.4}$$

Hint for (0.0.3)**:** The inequality (0.0.3) is easy when $w = 0$, so you may assume that $w \neq 0$. Define the function $q(t) \overset{\text{def}}{=} \langle v + tw, v + tw \rangle$. Using the properties of $\langle , \rangle$, show that $q(t) = \|v\|^2 + 2t\langle v, w \rangle + t^2\|w\|^2$ (hence q stands for "quadratic") and that $q(t) \geq 0$ for all t. Then using ordinary calculus, show that the minimum value taken by q occurs when $t_* = -\frac{\langle v,w\rangle}{\|w\|^2}$, and that the non-negativity of $q(t_*)$ implies the inequality (0.0.3).

IV. Let $f, g \in L^2(\mathbb{R})$ (i.e., $\int_{\mathbb{R}} |f(x)|^2 dx < \infty$, and similarly for g). Define $\langle f, g\rangle$ by

$$\langle f, g\rangle \overset{\text{def}}{=} \int_{\mathbb{R}} f(x)g(x)dx. \tag{0.0.5}$$

Show that the function $\langle , \rangle$ has all three properties of a dot product from the previous problem (for the second property, if you are unfamiliar with measure theory, then you are allowed to cheat a bit by assuming that the functions are continuous). Then use this to conclude the following super-important "Cauchy-Schwarz" inequality for integrals:

$$\left| \int_{\mathbb{R}} f(x)g(x)\,dx \right| \leq \left(\int_{\mathbb{R}} |f(x)|^2\,dx \right)^{1/2} \left(\int_{\mathbb{R}} |g(x)|^2\,dx \right)^{1/2}. \tag{0.0.6}$$

Remark 0.0.1. Inequalities analogous to (0.0.6) are used all the time in PDE analysis. An analogous inequality also holds if f and g are functions defined on a domain $\Omega \subset \mathbb{R}^n$ and the integrals are taken over Ω. Also, the Cauchy-Schwarz inequality holds if f and g are complex-valued functions.

Finally, show that if $f \in L^2(\mathbb{R})$, then

$$\left| \int_{\mathbb{R}} \sin(x)\frac{\ln(x^2+1)}{|x|^{3/4}} f(x)\,dx \right| \leq C \left(\int_{\mathbb{R}} |f(x)|^2\,dx \right)^{1/2}, \tag{0.0.7}$$

where $C > 0$ is some constant that *you do not have to explicitly evaluate.*

V. Read **Appendix A** of your textbook.

18.152 Introduction to Partial Differential Equations.
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