MATH 18.152 - PROBLEM SET 2

18.152 Introduction to PDEs, Fall 2011	Professor: Jared Speck
Problem Set 2, **Due: at the start of class on 9-22-11**	

I. Problem **2.1** on pg. 97.

II. Problem **2.3** on pg. 97 (the book forgot to tell you to set $L = \pi$ and $U = 0$).

III. Consider the solution $u(t,x) = \sum_{m=1}^{\infty}(-1)^{m+1}e^{-m^2\pi^2 t}\frac{2}{m\pi}\sin(m\pi x)$ to the initial-boundary value heat equation problem

$$\begin{cases} \partial_t u - \partial_x^2 u = 0, & (t,x) \in (0,\infty)\times(0,1), \\ u(0,x) = x, & x \in [0,1], \\ u(t,0) = u(t,1) = 0, & t \in (0,\infty), \end{cases}$$

as discussed in class. Show that

$$\lim_{t\downarrow 0}\|u(t,x) - x\|_{L^2([0,1])} = 0, \tag{0.1}$$

where the L^2 norm is taken over the x variable only. Feel free to make use of the "Some basic facts from Fourier analysis" theorem discussed in class.

Remark 0.0.1. This problem shows that even though there is a pointwise discontinuity at $(0,1)$, the solution is nonetheless "continuous in t at $t = 0$" with respect to the $L^2([0,1])$ spatial norm.

IV. Let $\ell > 0$ be a positive real number. Let $S \overset{\text{def}}{=} (0,\infty)\times(0,\ell)$, and let $u(t,x) \in C^{1,2}(\overline{S})$ be the solution of the initial-boundary value problem

$$\begin{cases} \partial_t u - \partial_x^2 u = 0, & (t,x) \in S, \\ u(0,x) = \ell^{-2}x(\ell - x), & x \in [0,\ell], \\ u(t,0) = 0, \qquad u(t,\ell) = 0, & t \in (0,\infty). \end{cases} \tag{0.2}$$

In this problem, you will use the energy method to show that the spatial L^2 norm of u decays exponentially *without actually having to solve the PDE.*

First show that $\|u(0,\cdot)\|_{L^2([0,\ell])} = \sqrt{\frac{\ell}{30}}$. Here, the notation $\|u(0,\cdot)\|_{L^2([0,\ell])}$ is meant to emphasize that the L^2 norm is taken over the spatial variable x only.

Next, show that $\frac{d}{dt}\big(\|u(t,\cdot)\|^2_{L^2([0,\ell])}\big) = -2\|\partial_x u(t,\cdot)\|^2_{L^2([0,\ell])}$.

Then show that $\|u(t,\cdot)\|_{C^0([0,\ell])} \leq \sqrt{\ell}\|\partial_x u(t,\cdot)\|_{L^2([0,\ell])}$ (**Hint**: Use the Fundamental theorem of calculus in x and the Cauchy-Schwarz inequality with one of the functions equal to 1.).

Then conclude that $\|u(t,\cdot)\|^2_{L^2} \leq \ell^2\|\partial_x u(t,\cdot)\|^2_{L^2([0,\ell])}$. Using a previous part of this problem, we conclude that $\frac{d}{dt}\big(\|u(t,\cdot)\|^2_{L^2([0,1])}\big) \leq -2\frac{1}{\ell^2}\|u(t,\cdot)\|^2_{L^2([0,\ell])}$.

Finally, integrate this differential inequality in time and use the initial conditions at $t = 0$ to conclude that $\|u(t,\cdot)\|_{L^2([0,1])} \leq \sqrt{\frac{\ell}{30}} e^{-t\ell^{-2}}$ for all $t \geq 0$.

V. In this problem, you will derive a very important solution to the heat equation on $\mathbb{R}^{1+1}$:

$$\partial_t u - D\partial_x^2 u = 0, \qquad (t,x) \in (0,\infty) \times \mathbb{R}. \tag{0.3}$$

The special solution $u(t,x)$ will be known as the *fundamental solution*, and it plays a very important role in the theory of the heat equation on $(0,\infty)\times\mathbb{R}$. We demand that our fundamental solution $u(t,x)$ should have the following properties:

- $u(t,x) \geq 0$
- $\int_{\mathbb{R}} u(t,x)\,dx = 1$ for all $t > 0$
- $\lim_{x\to\pm\infty} u(t,x) = 0$ for all $t > 0$
- $u(t,x) = u(t,-x)$ for all $t > 0$

To see that such a solution exists, first make the assumption that $u(t,x) = \frac{1}{\sqrt{Dt}} V(\zeta)$, where $\zeta \overset{\text{def}}{=} \frac{x}{\sqrt{Dt}}$ and $V(\zeta)$ is a function that is (hopefully) defined for all $\zeta \in \mathbb{R}$; we will motivate this assumption in class. Show that if u verifies (0.3), then V must satisfy the ODE

$$\frac{d}{d\zeta}(V'(\zeta) + \frac{1}{2}\zeta V(\zeta)) = 0. \tag{0.4}$$

Then, using the above demands, argue that $V(\zeta) = V(-\zeta)$, $V'(0) = 0$, and $\lim_{\zeta\to\pm} V(\zeta) = 0$.

Also using (0.4), argue that

$$V'(\zeta) + \frac{1}{2}\zeta V(\zeta) = 0. \tag{0.5}$$

Integrate (0.5) to conclude that $V(\zeta) = V(0)e^{-\frac{1}{4}\zeta^2}$, which implies that

$$u(t,x) = \frac{1}{\sqrt{Dt}} V(0) e^{-\frac{x^2}{4Dt}}. \tag{0.6}$$

Finally, use the second demand from above to conclude that $V(0) = \frac{1}{\sqrt{4\pi}}$.