MATH 18.152 - PROBLEM SET 3

18.152 Introduction to PDEs, Fall 2011 Professor: Jared Speck

Problem Set 3, **Due: at the start of class on 9-29-11**

I. Let $S \stackrel{\text{def}}{=} (0, \infty) \times (0, 1)$, and let $u \in C^{1,2}(\overline{S})$ be the solution of the initial-boundary value problem

$$\begin{cases} \partial_t u - \partial_x^2 u = 0, & (t, x) \in (0, \infty) \times (0, 1), \\ u(0, x) = x(1 - x), & x \in [0, 1], \\ u(t, 0) = 0, \qquad u(t, 1) = 0, & t \in (0, \infty). \end{cases}$$

Show that $u(t, x) = u(t, 1 - x)$ holds for all $t \geq 0$ and all $x \in [0, 1]$.

II. Let $S \stackrel{\text{def}}{=} (0, \infty) \times (0, 1)$, and let $u \in C^{1,2}(S) \cap C(\overline{S})$ be the solution of the initial-boundary value problem

$$\begin{cases} \partial_t u - \partial_x^2 u = 0, & (t, x) \in S, \\ u(0, x) = x(1 - x), & x \in [0, 1], \\ u(t, 0) = u(t, 1) = \text{some constant}, & t \in (0, \infty). \end{cases}$$

First prove that $u(t, x) \geq 0$ for $(t, x) \in \overline{S}$ (all the information you need to prove this has been given to you). Then find all numbers $\alpha > 0$ and $\beta > 0$ such that the following inequality holds on S :

$$u(t, x) \leq w(t, x) \stackrel{\text{def}}{=} \alpha x(1 - x)e^{-\beta t}. \tag{0.0.1}$$

Hint: Apply the maximum principle twice, once for u, and again for $w - u$.
Finally, use (0.0.1) to deduce that $u(t, x) \to 0$ as $t \to 0$ and that the convergence is *uniform* in x for $x \in [0, 1]$.

III. Problem **2.13** pg. 99.

IV. Problem **2.14** pg. 99. You can interpret the word "explicit" to mean that your solution is allowed to involve the standard fundamental solution for the heat equation.

V. In this problem you will consider PDEs on the set $(t, x) \in [0, \infty) \times \mathbb{R}^n$. You may assume that all of the functions involved are sufficiently differentiable. Let $\mathcal{L}$ be a linear differential operator of the form $\mathcal{L} = \partial_t + \widetilde{\mathcal{L}}$, where $\widetilde{\mathcal{L}}$ is a linear differential operator acting on the spatial variables alone (for example, $\widetilde{\mathcal{L}} = -\Delta$, as in the heat equation). Suppose that we know how to solve all homogeneous initial value problems

$$\begin{aligned} \mathcal{L}u(t, x) &= 0, && (t, x) \in [0, \infty) \times \mathbb{R}^n \\ u(0, x) &= f(t, x) && x \in \mathbb{R}^n. \end{aligned} \tag{0.0.2}$$

Suppose that we now want to solve the corresponding inhomogeneous problem with 0 data:

$$\begin{aligned} \mathcal{L}v(t,x) &= f(t,x), && (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ v(0,x) &= 0 && x \in \mathbb{R}^n. \end{aligned} \tag{0.0.3}$$

Show that a solution to (0.0.3) is

$$v(t,x) = \int_{s=0}^{t} v_{(s)}(t-s,x)\,ds, \tag{0.0.4}$$

where each $v_{(s)}$ is the solution to the following homogeneous initial value problem:

$$\begin{aligned} \mathcal{L}v_{(s)}(t,x) &= 0, && (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ v_{(s)}(0,x) &= f(s,x), && x \in \mathbb{R}^n. \end{aligned} \tag{0.0.5}$$

Note that in the PDE (0.0.5), s plays the role of a constant, whereas in equation (0.0.4), it plays the role of an integration variable. The formula (0.0.4) is known as **Duhamel's principle**.

Hint: Try directly showing that $v(t,x)$ has the correct initial condition and that it solves (0.0.3). You will have to make use of the fundamental theorem of calculus, and also differentiation under the integral; you may assume that the conditions necessary for differentiating under the integral are satisfied.

VI. Use the previous problem to show that a solution to the inhomogeneous heat equation

$$\begin{aligned} \partial_t u - D\Delta u &= f(t,x), && (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ u(0,x) &= g(x), && x \in \mathbb{R}^n \end{aligned} \tag{0.0.6}$$

is

$$u(t,x) = (\Gamma_D(t,\cdot) * g(\cdot))(x) + \int_0^t (\Gamma_D(t-s,\cdot) * f(s,\cdot))(x)\,ds, \tag{0.0.7}$$

where $\Gamma_D(t,x)$ is the fundamental solution introduced in class. Assume whatever conditions you need on f and g in order for your solution to be well-defined.

Hint: Show that the solution to (0.0.6) can be split into two pieces: $u = u_{hom} + u_{inhom}$, where u_{hom} solves the homogeneous heat equation with correct data $u_{hom}(0,x) = g(x)$, and u_{inhom} solves the correct equation $\partial_t u_{inhom} - D\Delta u_{inhom} = f(t,x)$ with data $u_{imhom}(0,x) = 0$. Then handle each piece separately. *This is a very standard technique for dissecting the solution of an inhomogeneous linear PDE into two parts.*

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