MATH 18.152 - PROBLEM SET 6

18.152 Introduction to PDEs, Fall 2011 Professor: Jared Speck

Problem Set 6, **Due: at the start of class on 10-20-11**

I. Consider the global Cauchy problem for the wave equation in $\mathbb{R}^{1+n}$:

$$-\partial_t^2 u(t,x) + \Delta u(t,x) = 0, \qquad (t,x) \in [0,\infty) \times \mathbb{R}^n, \tag{0.0.1a}$$

$$u(0,x) = f(x), \tag{0.0.1b}$$

$$\partial_t u(0,x) = g(x). \tag{0.0.1c}$$

Let the vectorfield $\mathbf{J}(t,x)$ on $\mathbb{R}^{1+n}$ be defined as follows:

$$\mathbf{J} = (J^0, J^1, \cdots, J^n) \stackrel{\text{def}}{=} \Big(\frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2, -\partial_1 u \partial_t u, -\partial_2 u \partial_t u, \cdots, -\partial_n u \partial_t u\Big). \tag{0.0.2}$$

Above, $x = (x^1, \cdots, x^n)$ denotes coordinates on $\mathbb{R}^n$, $\nabla u \stackrel{\text{def}}{=} (\partial_1 u, \cdots, \partial_n u)$ is the spatial gradient of u, and $|\nabla u|^2 \stackrel{\text{def}}{=} \sum_{i=1}^n (\partial_i u)^2$ is the square of its Euclidean length.

a) First show that

$$\partial_t J^0 + \sum_{i=1}^n \partial_i J^i = 0 \tag{0.0.3}$$

whenever u is a C^2 solution to (0.0.1a).

b) Then show that if $\mathbf{V} = (V^0, V^1, \cdots, V^n) = (1, \omega^1, \omega^2, \cdots, \omega^n) \in \mathbb{R}^{1+n}$ is any vector with $\sum_{i=1}^n (\omega_i)^2 \leq 1$, then

$$\mathbf{V} \cdot \mathbf{J} \stackrel{\text{def}}{=} \sum_{\mu=0}^n J^\mu V^\mu \geq 0. \tag{0.0.4}$$

Hint: To get started, try using the Cauchy-Schwarz inequality for dot products.

II. Assume that $0 \leq t \leq R$, and let $p \in \mathbb{R}^n$ be a fixed point. Let $\mathcal{C}_{t,p;R} \stackrel{\text{def}}{=} \{(\tau, y) \in [0,t) \times \mathbb{R}^n \mid |y - p| \leq R - \tau\} \subset \mathbb{R}^{1+n}$ be a solid, truncated backwards light cone. Note that the boundary of the cone consists of 3 pieces: $\partial\mathcal{C}_{t,p;R} = \mathcal{B} \cup \mathcal{M}_{t,p;R} \cup \mathcal{T}$, where $\mathcal{B} \stackrel{\text{def}}{=} \{0\} \times B_R(p)$ is the flat base of the truncated cone, $\mathcal{T} \stackrel{\text{def}}{=} \{t\} \times B_{R-t}(p)$ is the flat top of the truncated cone, and $\mathcal{M}_{t,p;R} \stackrel{\text{def}}{=} \{(\tau, y) \in [0,t) \times \mathbb{R}^n \mid |y - p| = R - \tau\}$ is the mantle (i.e., the side boundary) of the truncated cone.

Define the energy of a function u at time t on the **solid** ball $B_{R-t}(p)$ by

$$E^2(t;R;p) \stackrel{\text{def}}{=} \int_{B_{R-t}(p)} J^0(t,x)\, d^n x \stackrel{\text{def}}{=} \frac{1}{2}\int_{B_{R-t}(p)} (\partial_t u)^2 + |\nabla u|^2 \, d^n x, \tag{0.0.5}$$

and recall that the divergence theorem in $\mathbb{R}^{1+n}$ implies that

$$\int_{\mathcal{C}_{t,p;R}} \left(\partial_t J^0 + \sum_{i=1}^n \partial_i J^i\right) d^n x dt = \int_{\mathcal{M}_{t,p;R}} \mathbf{N}(\sigma)\cdot \mathbf{J}\, d\sigma - \underbrace{\int_{B_R(p)} J^0\, d^n x}_{E^2(0;R;p)} + \underbrace{\int_{B_{R-t}(p)} J^0\, d^n x}_{E^2(t;R;p)}. \tag{0.0.6}$$

In (0.0.6), $\mathbf{N}(\sigma)$ is the unit outward normal to $\mathcal{M}_{t,p;R}$.

Remark 0.0.1. In the near future, we will discuss the geometry of Minkowski spacetime, which is intimately connected to the linear wave equation. Our study will lead to a geometrically motivated construction of the vectorfield $\mathbf{J}$ and the identity (0.0.6). Alternatively, the identity (0.0.6) could also be derived by multiplying both sides of equation (0.0.1a) by $-\partial_t u$, then integrating by parts and using the divergence theorem.

a) Show that the unit outward normal $\mathbf{N}(\sigma)$ to $\mathcal{M}_{t,p;R}$ is of the form

$$\mathbf{N}(\sigma) = \frac{1}{\sqrt{2}}\big(1, \omega^1(\sigma), \omega^2(\sigma), \cdots, \omega^n(\sigma)\big), \tag{0.0.7}$$

where $\sum_{i=1}^n (\omega^i)^2 = 1$. Note that by translational invariance, you may assume that $p = 0$.

b) With the help of Problem **I** and (0.0.6) - (0.0.7), show that if u is a C^2 solution to (0.0.1a), then

$$E^2(t;R;p) \leq E^2(0;R;p) \tag{0.0.8}$$

holds for all t with $0 \leq t \leq R$.

c) Then show that if the functions $f(x)$ and $g(x)$ from (0.0.1b) - (0.0.1c) are both smooth and vanish outside of the ball $B_{R_0}(p) \subset \mathbb{R}^n$, then at each time $t \geq 0$, the solution $u(t,x)$ to (0.0.1a) vanishes outside of the ball $B_{R_0+t}(p)$.

d) Finally, under the same assumptions on f and g, let $R \to \infty$ in (0.0.8) (and also use additional arguments) to show that the solution u to (0.0.1a) satisfies

$$\||\nabla_{t,x} u(t,\cdot)|\|_{L^2(\mathbb{R}^n)} = \||\nabla_{t,x} u(0,\cdot)|\|_{L^2(\mathbb{R}^n)} \overset{\text{def}}{=} \left(\int_{\mathbb{R}^n} |g(x)|^2 + |\nabla f(x)|^2\, d^n x\right)^{1/2}, \tag{0.0.9}$$

where $\nabla_{t,x} u = (\partial_t u, \partial_1, \cdots, \partial_n u)$ is the spacetime gradient of u, $|\nabla_{t,x} u| \overset{\text{def}}{=} \sqrt{(\partial_t u)^2 + (\partial_1 u)^2 + \cdots (\partial_n u)^2}$, and the L^2 norms in (0.0.9) are taken over the spatial variables only.

III. Let $R > 0$, and let $f(x)$, $g(x)$ be smooth functions on $\mathbb{R}$ that vanish outside of $B_R(0) \overset{\text{def}}{=} [-R, R]$. Let $u(t,x)$ be the corresponding unique solution to the following global Cauchy problem on $\mathbb{R}^{1+1}$:

$$-\partial_t^2 u(t,x) + \partial_x^2 u(t,x) = 0, \tag{0.0.10a}$$

$$u(0,x) = f(x), \tag{0.0.10b}$$

$$\partial_t u(0,x) = g(x). \tag{0.0.10c}$$

We define the following quantities:

$$P^2(t) \overset{\text{def}}{=} \int_{\mathbb{R}} \big(\partial_x u(t,x)\big)^2\, dx, \qquad \text{the potential energy} \tag{0.0.11a}$$

$$K^2(t) \overset{\text{def}}{=} \int_{\mathbb{R}} \big(\partial_t u(t,x)\big)^2\, dx, \qquad \text{the kinetic energy} \tag{0.0.11b}$$

$$E^2(t) \overset{\text{def}}{=} P^2(t) + K^2(t), \qquad \text{the total energy.} \tag{0.0.11c}$$

In Problem **II**, you used energy methods to prove that $E(t)$ is conserved: $E(t) = E(0)$ for all $t \geq 0$. Now show that if t is large enough, then $P^2(t) = K^2(t) = \frac{1}{2}E^2(t)$. This is called **the equipartitioning** of the energy.

Hint: Try expressing $P(t)$ and $K(t)$ in terms of the *null derivatives* $\partial_q u(t,x)$ and $\partial_s u(t,x)$ that we used in the proof of d'Alembert's formula. If you set up the calculations properly, then the desired equipartitioning result should boil down to proving that $\int_{\mathbb{R}}(\partial_q u(t,x))(\partial_s u(t,x))\,dx = 0$ for all large t. In order to prove this latter result, take a close look at the the expressions for $\partial_q u(t,x)$ and $\partial_s u(t,x)$ that we derived in terms of f, g during that proof, and make use of the assumptions on f, g.

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