MATH 18.152 - PROBLEM SET 8

18.152 Introduction to PDEs, Fall 2011	Professor: Jared Speck
Problem Set 8, **Due: at the start of class on 11-10-11**	

I. Consider the energy-momentum tensor corresponding to the linear wave equation: $T_{\mu\nu} \overset{\text{def}}{=} \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m_{\mu\nu}(m^{-1})^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi$, and assume that $|\nabla_{t,x}\phi| \overset{\text{def}}{=} \sqrt{(\partial_t\phi)^2 + \sum_{i=1}^{n}(\partial_i\phi)^2} \neq 0$. Here, $(m^{-1})^{\mu\nu} = \text{diag}(-1, 1, 1, \cdots, 1)$ is the standard Minkowski metric on $\mathbb{R}^{1+n}$. Let X, Y be future-directed timelike vectors (i.e., $m(X,X) < 0, m(Y,Y) < 0$, $X^0 > 0$, and $Y^0 > 0$). Show that

$$T(X,Y) \overset{\text{def}}{=} T_{\alpha\beta}X^\alpha Y^\beta > 0. \tag{0.0.1}$$

Hint: First show that if L and $\underline{L}$ are any pair of null vectors normalized by $m(L,\underline{L}) = -2$, then $T(L,L) \geq 0$, $T(\underline{L},\underline{L}) \geq 0$, $T(L,\underline{L}) \geq 0$, and that at least one of these three must be non-zero. To prove these facts, it might be helpful to supplement the vectors L and $\underline{L}$ with some vectors $e_{(1)}, \cdots, e_{(n-1)}$ in order to form a null frame $\mathcal{N} \overset{\text{def}}{=} \{L, \underline{L}, e_{(1)}, \cdots, e_{(n-1)}\}$; the calculations will be much easier to do relative to the basis $\mathcal{N}$ compared to the standard basis for $\mathbb{R}^{1+n}$. Recall that $\mathcal{N} \overset{\text{def}}{=} \{L, \underline{L}, e_{(1)}, \cdots, e_{(n-1)}\}$ is any basis for $\mathbb{R}^{1+n}$ such that $0 = m(L,L) = m(\underline{L},\underline{L}) = m(L, e_{(i)}) = m(\underline{L}, e_{(i)})$ for $1 \leq i \leq n-1$, such that $m(L,\underline{L}) = -2$, such that $m(e_{(i)}, e_{(j)}) = 1$ if $i = j$, and such that $m(e_{(i)}, e_{(j)}) = 0$ if $i \neq j$; as we discussed in class, given any null pair $L, \underline{L}$ normalized by $m(L,\underline{L}) = -2$, there exists such a null frame $\mathcal{N}$ containing L and $\underline{L}$. Recall also that $(m^{-1})^{\mu\nu} = -\frac{1}{2}L^\mu\underline{L}^\nu - \frac{1}{2}\underline{L}^\mu L^\nu + \not{m}^{\mu\nu}$, where $\not{m}^{\mu\nu}$ is positive definite on $\text{span}\{e_{(1)}, \cdots, e_{(n-1)}\}$, $\not{m}^{\mu\nu}$ vanishes on $\text{span}\{L, \underline{L}\}$, and $\not{m}(L, e_{(i)}) = \not{m}(\underline{L}, e_{(i)}) = 0$ for $1 \leq i \leq n-1$.

To tackle the case of general X and Y, use Problem **V** from last week.

Remark 0.0.1. Inequality (0.0.1) also holds if X, Y are past-directed timelike vectors (i.e., $m(X,X) < 0, m(Y,Y) < 0$, $X^0 < 0$, and $Y^0 < 0$).

II. Consider the *Morawetz vectorfield* $\overline{K}^\mu$ on R^{1+3} defined by

$$\overline{K}^0 = 1 + t^2 + (x^1)^2 + (x^2)^2 + (x^3)^2, \tag{0.0.2}$$

$$\overline{K}^j = 2tx^j, \qquad (j = 1,2,3). \tag{0.0.3}$$

a) Show that $\overline{K}$ is future-directed and timelike. Above, (t, x^1, x^2, x^3) are the standard coordinates on R^{1+3}.

b) Show that

$$\partial_\mu\overline{K}_\nu + \partial_\nu\overline{K}_\mu = 4tm_{\mu\nu}, \qquad (\mu,\nu = 0,1,2,3), \tag{0.0.4}$$

where $m_{\mu\nu}$ denotes the Minkowski metric.

Remark 0.0.2. $\overline{K}$ is said to be a *conformal Killing field* of the Minkowski metric because the right-hand side of (0.0.4) is proportional to $m_{\mu\nu}$.

c) Show that

$$m_{\mu\nu}T^{\mu\nu} = 0, \tag{0.0.5}$$

where $T^{\mu\nu} \overset{\text{def}}{=} (m^{-1})^{\mu\alpha}(m^{-1})^{\nu\beta}T_{\alpha\beta}$ is the energy-momentum tensor from Problem **I** with its indices raised. Note that the formula (0.0.5) only holds in $1+3$ spacetime dimensions.

d) Show that $\partial_\mu {}^{(\overline{K})}J^\mu = 0$ whenever ϕ is a solution to the linear wave equation $(m^{-1})^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 0$, where

$${}^{(\overline{K})}J^\mu \overset{\text{def}}{=} -T^{\mu\nu}\overline{K}_\nu. \tag{0.0.6}$$

e) Show that

$${}^{(\overline{K})}J^0 = \frac{1}{4}\Big\{\big[1+(t+r)^2\big](\nabla_L\phi)^2 + \big[1+(t-r)^2\big](\nabla_{\underline{L}}\phi)^2 + 2\big[1+t^2+r^2\big]\ {\not}m^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Big\}. \tag{0.0.7}$$

Above, $(m^{-1})^{\mu\nu} = -\frac{1}{2}L^\mu\underline{L}^\nu - \frac{1}{2}\underline{L}^\mu L^\nu + {\not}m^{\mu\nu}$ is the standard null decomposition of $(m^{-1})^{\mu\nu}$ from class. In particular, $L^\mu = (1, \frac{x^1}{r}, \frac{x^2}{r}, \frac{x^3}{r})$, $\underline{L}^\mu = (1, -\frac{x^1}{r}, -\frac{x^2}{r}, -\frac{x^3}{r})$, $\nabla_L\phi = \partial_t\phi + \partial_r\phi$, $\nabla_{\underline{L}}\phi = \partial_t\phi - \partial_r\phi$, and ${\not}m^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ is the square of the Euclidean norm of the *angular* derivatives of ϕ. Here, $r \overset{\text{def}}{=} \sqrt{(x^1)^2+(x^2)^2+(x^3)^2}$ denotes the standard spherical coordinate on $\mathbb{R}^3$, and ∂_r denotes the standard radial derivative.
Hint: The following expansions in terms of L and $\underline{L}$ may be very helpful:

(0.0.8)
$$\overline{K}^\mu = \frac{1}{2}\Big\{[1+(r+t)^2]L^\mu + [1+(r-t)^2]\underline{L}^\mu\Big\},$$
(0.0.9)
$$(1,0,0,0) = \frac{1}{2}(L^\mu + \underline{L}^\nu),$$
(0.0.10)
$$\begin{aligned}{}^{(\overline{K})}J^0 &= T\big(\overline{K}, \frac{1}{2}(L+\underline{L})\big)\\ &= \frac{1}{4}\Big\{[1+(r+t)^2]T\big(L,L\big) + [1+(r-t)^2]T\big(\underline{L},\underline{L}\big) + \big([1+(r+t)^2]+[1+(r-t)^2]\big)T\big(L,\underline{L}\big)\Big\}.\end{aligned}$$

f) Finally, with the help of the vectorfield ${}^{(\overline{K})}J^\mu$, apply the divergence theorem on an appropriately chosen spacetime region and use the previous results to derive the following conservation law for smooth solutions to the linear wave equation $(m^{-1})^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 0:$

(0.0.11)

$$\int_{\mathbb{R}^3} \frac{1}{4}\Big\{ \big[1 + (t+r)^2\big](\nabla_L \phi(t,x))^2 + \big[1 + (t-r)^2\big](\nabla_{\underline{L}} \phi(t,x))^2 + 2\big[1 + t^2 + r^2\big] \not{m}^{\mu\nu} \partial_\mu \phi(t,x) \partial_\nu \phi(t,x) \Big\} \, d^3x$$
$$= \int_{\mathbb{R}^3} \frac{1}{4}\Big\{ \big[1 + r^2\big](\nabla_L \phi(0,x))^2 + \big[1 + r^2\big](\nabla_{\underline{L}} \phi(0,x))^2 + 2\big[1 + r^2\big] \not{m}^{\mu\nu} \partial_\mu \phi(0,x) \partial_\nu \phi(0,x) \Big\} \, d^3x.$$

For simplicity, at each fixed t, you may assume that there exists an $R > 0$ such that $\phi(t,x)$ vanishes whenever $|x| \geq R$.

Remark 0.0.3. Note that the right-hand side of (0.0.11) can be computed in terms of the initial data alone. Note also that the different *null derivatives* of ϕ appearing on the left-hand side of (0.0.11) carry different weights. In particular, $\nabla_L \phi$ and the angular derivatives of ϕ have larger weights than $\nabla_{\underline{L}} \phi$. These larger weights are strongly connected to the following fact, whose full proof requires additional methods going beyond this course: $\nabla_L \phi$ and the angular derivatives of ϕ decay faster in t compared to $\nabla_{\underline{L}} \phi$.

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Fall 2011