MATH 18.152 - PROBLEM SET 9

18.152 Introduction to PDEs, Fall 2011	Professor: Jared Speck
Problem Set 9, **Due: at the start of class on 11-17-11**	

I. Classify the following PDEs as elliptic, hyperbolic, or parabolic.

a)

$$\partial_t^2 u + \partial_t \partial_x u + \partial_x^2 u = 0 \tag{0.0.1}$$

b)

$$\partial_t^2 u + 2\partial_t \partial_x u + \partial_x^2 u = 0 \tag{0.0.2}$$

c)

$$2\partial_t^2 u - \partial_t \partial_x u - 12\partial_x^2 u = 0 \tag{0.0.3}$$

II. Consider the function $\mathrm{sinc} : \mathbb{R} \to \mathbb{R}$ defined by

$$\mathrm{sinc}(x) \stackrel{\mathrm{def}}{=} \begin{cases} \frac{\sin \pi x}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases} \tag{0.0.4}$$

a) Show that $\mathrm{sinc}(x)$ is infinitely differentiable at all points $x \in \mathbb{R}$.
Hint: Taylor series.

b) Let $a > 0$ be a constant, and let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) \stackrel{\mathrm{def}}{=} \begin{cases} 1, & |x| \leq a, \\ 0, & |x| > a. \end{cases} \tag{0.0.5}$$

f is sometimes referred to as the *characteristic* function of the interval $[-a, a]$. Show that

$$\hat{f}(\xi) = 2a\mathrm{sinc}(2a\xi). \tag{0.0.6}$$

III. Let $C_0(\mathbb{R})$ denote the set of all continuous function $u : \mathbb{R} \to \mathbb{C}$ such that $\lim_{x \to \pm\infty} u(x) = 0$. Let $\| \cdot \|_{C_0}$ be the norm on $C_0(\mathbb{R})$ defined by $\|u\|_{C_0} \stackrel{\mathrm{def}}{=} \max_{x \in \mathbb{R}} |u(x)|$. Let $\{u_j\}_{j=1}^{\infty}$ be a sequence of functions such that $u_j \in C_0(\mathbb{R})$ for $1 \leq j$, and let $u : \mathbb{R} \to \mathbb{C}$ be a function such that

$$\lim_{n \to \infty} \|u - u_n\|_{C_0} = 0. \tag{0.0.7}$$

From standard real analysis, it follows that u is a continuous function since it is the uniform limit of continuous functions. Show that in addition, we have $u \in C_0(\mathbb{R})$, i.e. that $\lim_{x\to\pm\infty} u(x) = 0$.

IV. Let $R > 0$ be a real number, and let $f : \mathbb{R} \to \mathbb{C}$ be a complex-valued, continuous function that vanishes when $|x| \geq R$. For each $y \in \mathbb{R}$, let $\tau_y f$ be the translate of f by y, i.e.,

$$\tau_y f(x) \overset{\text{def}}{=} f(x - y). \tag{0.0.8}$$

Show that

$$\lim_{y\to 0} \|\tau_y f - f\|_{L^1} \overset{\text{def}}{=} \lim_{y\to 0} \int_{\mathbb{R}} |f(x - y) - f(x)|\, dx = 0. \tag{0.0.9}$$

Hint: Use the fact that f is uniformly continuous on all of $\mathbb{R}$ and that for all small y, the integrand in (0.0.9) vanishes outside of fixed compact set of x values.

V. Let $R > 0$ be a real number, and let $f : \mathbb{R} \to \mathbb{C}$ be a complex-valued, continuous function that vanishes when $|x| \geq R$. For any $t > 0$, define the function f_t by

$$f_t(x) \overset{\text{def}}{=} (\Gamma(t, \cdot) * f(\cdot))(x), \tag{0.0.10}$$

where $*$ denotes convolution, and

$$\Gamma(t, x) \overset{\text{def}}{=} \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)} \tag{0.0.11}$$

is the fundamental solution to the heat equation with diffusion constant $D = 1$.

Show that

$$\lim_{t\downarrow 0} \|f_t - f\|_{L^1} = 0. \tag{0.0.12}$$

Hint: Using the fact that $\int_{\mathbb{R}} \Gamma(t, y)\, dy = 1$ for all $t > 0$, it is easy to see that (0.0.12) is equivalent to proving that

$$\lim_{t\downarrow 0} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \Gamma(t, y) \Big\{ f(x - y) - f(x) \Big\}\, dy \right| dx = 0. \tag{0.0.13}$$

With the help of Fubini's theorem, to prove (0.0.13), it suffices to show that

$$\lim_{t\downarrow 0} \int_{\mathbb{R}} |\Gamma(t, y)| \int_{\mathbb{R}} \Big| f(x - y) - f(x) \Big|\, dx\, dy = 0. \tag{0.0.14}$$

To prove (0.0.14), it is helpful to split the y integral into two pieces: one over a small ball $B_r(0) \overset{\text{def}}{=} \{y \mid |y| \leq r\}$, and the second one over its complement.

To show that the integral

$$\int_{B_r(0)} \Gamma(t, y) \int_{\mathbb{R}} \Big| f(x - y) - f(x) \Big|\, dx\, dy \tag{0.0.15}$$

is small for an appropriate choice of r, use the properties of $\Gamma(t, y)$ and Problem **IV**.

For a fixed r, in order to show that the complementary integral

$$\int_{\{y \mid |y| \geq r\}} \Gamma(t,y) \int_{\mathbb{R}} \Big| f(x-y) - f(x) \Big| \, dx \, dy, \tag{0.0.16}$$

is small for sufficiently small t, use the fact that $\Gamma(t,y)$ rapidly decays as a function of y when t is small and the triangle inequality-type estimate $\int_{\mathbb{R}} \Big| f(x-y) - f(x) \Big| \, dx \leq 2\|f\|_{L^1}$ (independently of y).

VI. Let $R > 0$ be a real number, and let $f : \mathbb{R} \to \mathbb{C}$ be a complex-valued, continuous function that vanishes when $|x| \geq R$. For any $t > 0$, define

$$f_t(x) \stackrel{\text{def}}{=} (\Gamma(t,\cdot) * f(\cdot))(x), \tag{0.0.17}$$

where $*$ denotes convolution, and

$$\Gamma(t,x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)} \tag{0.0.18}$$

is the fundamental solution to the heat equation with diffusion constant $D = 1$.

a) Using the basic properties of the Fourier transform, show that for any $t > 0$,

$$\hat{f}_t(\xi) = \hat{f}(\xi) e^{-4t\pi^2|\xi|^2}. \tag{0.0.19}$$

b) Using the basic properties of the Fourier transform, show that for any $t > 0$, $\hat{f}_t$ and $\hat{f}$ are continuous, *bounded* functions of ξ and that

$$\|\hat{f}_t - \hat{f}\|_{C_0} \leq \|f_t - f\|_{L^1(\mathbb{R})}. \tag{0.0.20}$$

Above, the norm $\|\cdot\|_{C_0}$ is as in Problem **III**.

c) Using parts **a)** and **b)**, show that for any $t > 0$, $\hat{f}_t \in C_0(\mathbb{R})$, where the function space $C_0(\mathbb{R})$ is defined in Problem **III**.

d) Using part **b)** and Problem **V**, show that

$$\lim_{t \downarrow 0} \|\hat{f}_t - \hat{f}\|_{C_0} = 0. \tag{0.0.21}$$

e) Using part **c)**, part **d)**, and Problem **III**, show that $\hat{f} \in C_0(\mathbb{R})$, and therefore (by definition) that

$$\lim_{\xi \to \pm\infty} \hat{f}(\xi) = 0. \tag{0.0.22}$$

Remark 0.0.1. (0.0.22) is a version of the Riemann-Lebesgue lemma. Written out in full form, it states that

$$\lim_{\xi \to \pm\infty} \int_{\mathbb{R}} e^{-2\pi i \xi x} f(x) \, dx = 0. \tag{0.0.23}$$

With slightly more advanced techniques, the above hypotheses on f can be weakened to only the assumption $f \in L^1$.

18.152 Introduction to Partial Differential Equations.
Fall 2011