MATH 18.152 COURSE NOTES - CLASS MEETING # 3

18.152 Introduction to PDEs, Fall 2011 Professor: Jared Speck

Class Meeting # 3: The Heat Equation: Uniqueness

1. Uniqueness

The results from the previous lecture produced one solution to the Dirichlet problem

(1.0.1)
$$\begin{cases} u_t - u_{xx} = 0, & (t,x) \in (0,T] \times [0,1], \\ u(0,x) = x, & x \in [0,1], \\ u(t,0) = 0, & u(t,1) = 0, \end{cases}$$

namely

(1.0.2)
$$u(t,x) = \sum_{m=1}^{\infty} (-1)^{m+1} e^{-m^2\pi^2 t} \frac{2}{m\pi} \sin(m\pi x).$$

But how do we know that this is the only one? In other words, we need to answer the uniqueness question (6) from the previous lecture. The next theorem addresses this question. We first need to introduce some important spacetime domains that will play a role in the analysis.

Definition 1.0.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded spatial domain (i.e., an open connected subset of $\mathbb{R}^n$), and let $T > 0$ be a time. We define the corresponding *spacetime cylinder* $Q_T \subset \mathbb{R}^{1+n}$ by

(1.0.3)
$$Q_T \overset{\text{def}}{=} (0,T) \times \Omega.$$

We also define the *parabolic boundary* $\partial_p Q_T$ of Q_T as follows:

(1.0.4)
$$\begin{aligned} \partial_p Q_T &\overset{\text{def}}{=} \{0\} \times \overline{\Omega} \cup (0,T] \times \partial\Omega \\ &= \text{bottom of } \overline{Q}_T \cup \text{ sides of } \overline{Q}_T. \end{aligned}$$

Here, $\overline{Q}_T$ denotes the closure of Q_T in $\mathbb{R}^{1+n}$.

Theorem 1.1 (**A uniqueness result for the heat equation on a finite interval**). *Solutions $u \in C^{1,2}(\overline{Q}_T)$ to the inhomogeneous heat equation*

(1.0.5)
$$\partial_t u - D\partial_x^2 u = f(t,x)$$

*are **unique** under Dirichlet, Neumann, Robin, or mixed conditions.*

Remark 1.0.1. By $u \in C^{1,2}(\overline{Q}_T)$, we mean that the time derivatives of $u(t,x)$ up to order 1 (the first index) are continuously differentiable on Q_T and extend continuously to the closure of Q_T, and also that all spatial derivatives of $u(t,x)$ up to order 2 (the second index) are continuously differentiable on Q_T and extend continuously to the closure of Q_T. Unfortunately, these kind of ugly technical details often play a role in PDE theory.

Remark 1.0.2. In its current form Theorem, 1.1 is not quite strong enough to apply to the problem (1.0.1). More precisely, the solution to that problem has a discontinuity at $(0, 1)$, while Theorem 1.1 requires that the solutions are of class $C^{1,2}(\overline{Q}_T)$. Uniqueness does in fact hold in a certain sense for the problem (1.0.1), but the because of the discontinuity, this issue is best addressed in a more advanced course.

Proof. Let's do the Dirichlet proof in the case $D = 1$. Assume we have two solutions to (1.0.5) with specified Cauchy and Dirichlet data. Then by subtracting them and calling the difference w, we get another solution w satisfying

$$\begin{cases} \partial_t w - \partial_x^2 w = 0, & (t,x) \in [0,T] \times [0,L], \\ w(0,x) = 0, & x \in [0,L], \\ w(t,0) = 0, \quad w(t,L) = 0, & t \in [0,T]. \end{cases} \tag{1.0.6}$$

We want to show that $w(t,x) = 0$ for $(t,x) \in [0,T] \times [0,L]$. We perform the following super-important and very commonly used strategy: we multiply both sides of (1.0.6) by w and integrate dx over the interval $[0,L]$ to derive

$$\int_{[0,L]} w \partial_t w \, dx = \int_{[0,L]} w \partial_x^2 w \, dx \tag{1.0.7}$$

differentiate under the integral

$$\underbrace{\Longrightarrow} \quad \frac{d}{dt}\frac{1}{2}\int_{[0,L]} w^2(t,x)\, dx = \int_{[0,L]} w\partial_t w \, dx = \underbrace{\int_{[0,L]} w \partial_x^2 w\, dx}_{\text{integrate by parts}}$$

$$= \underbrace{- \int_{[0,L]} (\partial_x w(t,x))^2 \, dx}_{\leq 0} + \underbrace{w(t,x)\partial_x w(t,x)|_{x=0}^{x=L}}_{= 0 \text{ by bndry. cond.}}$$

$$\leq 0.$$

So if we define the *energy*

$$E(t) \stackrel{\text{def}}{=} \underbrace{\int_{[0,L]} w^2(t,x)\, dx}_{\geq 0}, \tag{1.0.8}$$

then we have shown that

$$\frac{d}{dt} E(t) \leq 0. \tag{1.0.9}$$

But $E(0) = 0$ by the initial conditions of w. Therefore, $E(t) = 0$ for $t \in [0,T]$. But since $w^2(t,x)$ is continuous and non-negative, it must be that $w^2(t,x) = 0$ for $(t,x) \in [0,T] \times [0,L]$.

□

Remark 1.0.3. Broadly speaking, the strategy we have used in this proof is called the *energy method*. It is a very flexible strategy that applies to many PDEs.

Note also that we did not need to know very much about the solution to conclude that it is unique! In particular, we didn't need to "find a formula" for the solution!

Note also that $E(t)$ is the square of the spatial $L^2([0, L])$ norm of w at time t.

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