MATH 18.152 COURSE NOTES - CLASS MEETING # 4

18.152 Introduction to PDEs, Fall 2011	Professor: Jared Speck
Class Meeting #4: The Heat Equation: The Weak Maximum Principle	

1. The Weak Maximum Principle

We will now study some important properties of solutions to the heat equation $\partial_t u - D\Delta u = 0$. For simplicity, we sometimes only study the case of $1+1$ spacetime dimensions, even though analogous properties are verified in higher dimensions.

Theorem 1.1 (Weak Maximum Principle). *Let $\Omega \subset \mathbb{R}^n$ be a domain. Recall that $Q_T \overset{def}{=} (0,T) \times \Omega$ is a spacetime cylinder and that $\partial_p Q_T \overset{def}{=} \{0\} \times \overline{\Omega} \cup (0,T] \times \partial\Omega$ is its corresponding parabolic boundary. Let $w \in C^{1,2}(Q_T) \cap C(\overline{Q}_T)$ be a solution to the (possibly inhomogeneous) heat equation*

$$w_t - D\Delta w = f, \tag{1.0.1}$$

where $f \leq 0$. Then $w(t,x)$ obtains its max in the region $\overline{Q}_T$ on $\partial_p Q_T$. Thus, if w is strictly negative on $\partial_p Q_T$, then w is strictly negative on $\overline{Q}_T$.

Proof. For simplicity, we consider only case of $1+1$ spacetime dimensions. Let ϵ be a positive number, and let $u = w - \epsilon t$. Our goal is to first study u, and then take a limit as $\epsilon \downarrow 0$ to extract information about w. Note that on $\overline{Q}_T$ we have $u \leq w$, that $w \leq u + \epsilon T$, and that in Q_T we have

$$u_t - Du_{xx} = f - \epsilon < 0. \tag{1.0.2}$$

We claim that the maximum of u on $\overline{Q}_{T-\epsilon}$ occurs on $\partial_p Q_{T-\epsilon}$. To verify the claim, suppose that $u(t,x)$ has its max at $(t_0,x_0) \in \overline{Q}_{T-\epsilon}$. We may assume that $0 < t_0 \leq T - \epsilon$, since if $t_0 = 0$ the claim is obviously true. Under this assumption, we have that $u < w$ and that $w \leq u + \epsilon T$. Similarly, we may also assume that $x \in \Omega$, since otherwise we would have $(t,x) \in \partial_p Q_{T-\epsilon}$, and the claim would be true.

Then from vector calculus, $u_x(t_0,x_0)$ must be equal to 0. Furthermore, $u_t(t_0,x_0)$ must also be equal to 0 if $t_0 < T - \epsilon$, and $u_t(t_0,x_0) \geq 0$ if $t_0 = T - \epsilon$. Now since $u(t_0,x_0)$ is a maximum value, we can apply Taylor's remainder theorem in x to deduce that for x near x_0, we have

$$u(t_0,x) - u(t_0,x_0) = \underbrace{u_x|_{t_0,x_0}(x-x_0)}_{0} + u_{xx}|_{t_0,x^*}(x-x_0)^2 \leq 0, \tag{1.0.3}$$

where x_* is some point in between x_0 and x. Therefore, $u_{xx}(t_0,x^*) \leq 0$, and by taking the limit as $x \to x_0$, it follows that $u_{xx}(t_0,x_0) \leq 0$. Thus, in any possible case, we have that

$$u_t(t_0,x_0) - Du_{xx}(t_0,x_0) \geq 0, \tag{1.0.4}$$

which contradicts (1.0.2).

Using $u \leq w$ and that fact that $\partial_p Q_{T-\epsilon} \subset \partial_p Q_T$, we have thus shown that

$$\max_{\overline{Q}_{T-\epsilon}} u = \max_{\partial_p Q_{T-\epsilon}} u \leq \max_{\partial_p Q_{T-\epsilon}} w \leq \max_{\partial_p Q_T} w. \tag{1.0.5}$$

Using (1.0.5) and $w \leq u + \epsilon T$, we also have that

$$\max_{\overline{Q}_{T-\epsilon}} w \leq \max_{\overline{Q}_{T-\epsilon}} u + \epsilon T \leq \epsilon T + \max_{\partial_p Q_T} w. \tag{1.0.6}$$

Now since w is uniformly continuous on $\overline{Q}_T$, we have that

$$\max_{\overline{Q}_{T-\epsilon}} w \uparrow \max_{\overline{Q}_T} w \tag{1.0.7}$$

as $\epsilon \downarrow 0$. Thus, allowing $\epsilon \downarrow 0$ in inequality (1.0.6), we deduce that

$$\max_{\overline{Q}_T} w = \lim_{\epsilon \downarrow 0} \max_{\overline{Q}_{T-\epsilon}} w \leq \lim_{\epsilon \downarrow 0} (\epsilon T + \max_{\partial_p Q_T} w) = \max_{\partial_p Q_T} w \leq \max_{\overline{Q}_T} w. \tag{1.0.8}$$

Therefore, all of the inequalities in (1.0.8) can be replaced with equalities, and

$$\max_{\overline{Q}_T} w = \max_{\partial_p Q_T} w \tag{1.0.9}$$

as desired.

□

The following very important corollary shows how to *compare* two different solutions to the heat equation with possibly different inhomogeneous terms. The proof relies upon the weak maximum principle.

Corollary 1.0.1 (Comparison Principle and Stability). *Suppose that v, w are solutions to the heat equations*

$$v_t - D v_{xx} = f, \tag{1.0.10}$$

$$w_t - D w_{xx} = g. \tag{1.0.11}$$

Then

1. *(Comparison): If $v \geq w$ on $\partial_p Q_T$ and $f \geq g$, then $v \geq w$ on all of Q_T.*
2. *(Stability):* $\max_{\overline{Q}_T} |v - w| \leq \max_{\partial_p Q_T} |v - w| + T \max_{\overline{Q}_T} |f - g|$.

Proof. One of the things that makes linear PDEs relatively easy to study is that you can add or subtract solutions: Setting $u \stackrel{\text{def}}{=} w - v$, we have

$$u_t - D u_{xx} = g - f \leq 0. \tag{1.0.12}$$

Then by Theorem 1.1, since $u \leq 0$ on $\partial_p Q_T$ we have that $u \leq 0$ on Q_T. This proves (1).

To prove (2), we define $M \stackrel{\text{def}}{=} \max_{\overline{Q}_T} |f - g|$, $u \stackrel{\text{def}}{=} w - v - tM$ and note that

$$u_t - Du_{xx} = g - f - M \leq 0. \tag{1.0.13}$$

Thus, by Theorem 1.1, we have that

$$\max_{\overline{Q}_T} u = \max_{\partial_p Q_T} u \leq \max_{\partial_p Q_T} |w - v|. \tag{1.0.14}$$

Thus, subtracting and adding tM, we have

$$\max_{\overline{Q}_T} w - v \leq \max_{\overline{Q}_T}(w - v - tM) + \max_{\overline{Q}_T} tM \leq \max_{\partial_p Q_T} |w - v| + TM. \tag{1.0.15}$$

Similarly, by setting $u \overset{\text{def}}{=} v - w - tM$, we can show that

$$\max_{\overline{Q}_T} v - w \leq \max_{\partial_p Q_T} |w - v| + TM. \tag{1.0.16}$$

Combining (1.0.15) and (1.0.16), and recalling the definition of M, we have shown (2).

□

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