MATH 18.152 COURSE NOTES - CLASS MEETING # 6

18.152 Introduction to PDEs, Fall 2011 Professor: Jared Speck

Class Meeting # 6: Laplace's and Poisson's Equations

We will now study the Laplace and Poisson equations on a domain (i.e. open connected subset) $\Omega \subset \mathbb{R}^n$. Recall that

$$\Delta \overset{\text{def}}{=} \sum_{i=1}^{n} \partial_i^2. \tag{0.0.1}$$

The Laplace equation is

$$\Delta u(x) = 0, \qquad x \in \Omega, \tag{0.0.2}$$

while the Poisson equation is the inhomogeneous equation

$$\Delta u(x) = f(x). \tag{0.0.3}$$

Functions $u \in C^2(\Omega)$ verifying (0.0.2) are said to be *harmonic*. (0.0.2) and (0.0.3) are both second order, linear, constant coefficient PDEs. As in our study of the heat equation, we will need to supply some kind of boundary conditions to get a well-posed problem. But unlike the heat equation, there is *no "timelike" variable, so there is no "initial condition" to specify*!

1. Where does it come from?

1.1. **Basic examples.** First example: set $\partial_t u \equiv 0$ in the heat equation, and (0.0.2) results. These solutions are known as *steady state solutions.*

Second example: We start with Maxwell equations from electrodynamics. The quantities of interest are

- $\mathbf{E} = \big(E_1(t,x,y,z), E_2(t,x,y,z), E_3(t,x,y,z)\big)$ is the *electric field*
- $\mathbf{B} = \big(B_1(t,x,y,z), B_2(t,x,y,z), B_3(t,x,y,z)\big)$ is the *magnetic induction*
- $\mathbf{J} = \big(J_1(t,x,y,z), J_2(t,x,y,z), J_3(t,x,y,z)\big)$ is the *current density*
- ρ is the *charge density*

Maxwell's equations are

$$\partial_t \mathbf{E} - \nabla \times \mathbf{B} = -\mathbf{J}, \qquad \nabla \cdot \mathbf{E} = \rho, \tag{1.1.1}$$

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{B} = 0. \tag{1.1.2}$$

Recall that $\nabla\times$ is the curl operator, so that e.g. $\nabla \times \mathbf{B} = \big(\partial_y B_3 - \partial_z B_2, \partial_z B_1 - \partial_x B_3, \partial_x B_2 - \partial_y B_1\big)$. Let's look for steady-state solutions with $\partial_t \mathbf{E} = \partial_t \mathbf{B} \equiv 0$. Then equation (1.1.2) implies that

$$\nabla \times \mathbf{E} = 0, \tag{1.1.3}$$

so that by the Poincaré lemma, there exists a scalar-valued function $\phi(x, y, z)$ such that

$$\mathbf{E}(x, y, z) = -\nabla\phi(x, y, z). \tag{1.1.4}$$

The function ϕ is called an *electric potential.* Plugging (1.1.4) into the second of (1.1.1), and using the identity $\nabla \cdot \nabla\phi = \Delta\phi$, we deduce that

$$\Delta\phi(x, y, z) = -\rho(x, y, z). \tag{1.1.5}$$

This is exactly the Poisson equation (0.0.3) with inhomogeneous term $f = -\rho$. Thus, Poisson's equation is at the heart of electrostatics.

1.2. **Connections to complex analysis.** Let $z = x + iy$ (where $x, y \in \mathbb{R}$) be a complex number, and let $f(z) = u(z) + iv(z)$ be a complex-valued function (where $u, v \in \mathbb{R}$). We recall that f is said to be differentiable at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{1.2.1}$$

exists. If the limit exists, we denote it by $f'(z_0)$.

A fundamental result of complex analysis is the following: f is differentiable at $z_0 = x_0 + iy_0 \simeq (x_0, y_0)$ if and only if the real and imaginary parts of f verify the *Cauchy-Riemann equations* at z_0 :

$$u_x(x_0, y_0) = v_y(x_0, y_0), \tag{1.2.2}$$

$$u_y(x_0, y_0) = -v_x(x_0, y_0). \tag{1.2.3}$$

Differentiating (1.2.2) and using the symmetry of mixed partial derivatives (we are assuming here that $u(x, y)$ and $v(x, y)$ are C^1 near (x_0, y_0)), we have

$$\Delta u \stackrel{\text{def}}{=} u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0, \tag{1.2.4}$$

$$\Delta v \stackrel{\text{def}}{=} v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0. \tag{1.2.5}$$

Thus, *the real and imaginary parts of a complex-differentiable function are harmonic*!

2. Well-posed Problems

Much like in the case of the heat equation, we are interested in well-posed problems for the Laplace and Poisson equations. Recall that well-posed problems are problems that i) have a solution; ii)the solutions are unique; and iii)the solution varies continuously with the data.

Let $\Omega \subset \mathbb{R}^n$ be a domain with a Lipschitz boundary, and let $\hat{N}$ denote the unit outward normal vector to $\partial\Omega$. We consider the PDE

$$\Delta u(x) = f(x), \qquad x \in \Omega, \tag{2.0.6}$$

supplemented by some boundary conditions. The following boundary conditions are known to lead to well-posed problems:

(1) Dirichlet data: specify a function $g(x)$ defined on $\partial\Omega$ such that $u|_{\partial\Omega}(x) = g(x)$.
(2) Neumann data: specify a function $h(x)$ defined on $\partial\Omega$ such that $\nabla_{\hat{N}} u(x)|_{\partial\Omega}(x) = h(x)$.
(3) Robin-type data: specify a function $h(x)$ defined on $\partial\Omega$ such that $\nabla_{\hat{N}} u(x)|_{\partial\Omega}(x) + \alpha u|_{\partial\Omega}(x) = h(x)$, where $\alpha > 0$ is a constant.
(4) Mixed conditions: for example, we can divide $\partial\Omega$ into two *disjoint* pieces $\partial\Omega = S_D \cup S_N$, where S_N is relatively open in $\partial\Omega$, and specify a function $g(x)$ defined on S_D and a function $h(x)$ defined on S_N such that $u|_{S_D}(x) = g(x)$, $\nabla_{\hat{N}} u|_{S_N}(x) = h(x)$.
(5) Conditions at infinity: When $\Omega = \mathbb{R}^n$, we can specify asymptotic conditions on $u(x)$ as $|x| \to \infty$. We will return to this kind of condition later in the course.

3. Uniqueness via the Energy Method

In this section, we address the question of uniqueness for solutions to the equation (0.0.3), supplemented by suitable boundary conditions. As in the case of the heat equation, we are able to provide a simple proof based on the energy method.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded domain. Then under Dirichlet, Robin, or mixed boundary conditions, there is at most one solution of regularity $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ to the Poisson equation* (0.0.3).

In the case of Neumann conditions, any two solutions can differ by at most a constant.

Proof. If u and v are two solutions to (0.0.3) with the same boundary data, then we can subtract them (aren't linear PDEs nice?!...) to get a solution $w \overset{\text{def}}{=} u - v$ to the Poisson equation with 0 data:

$$\Delta w = 0. \tag{3.0.7}$$

Let's perform the usual trick of multiplying (3.0.7) by w, integrating over Ω, and integrating by parts via the divergence theorem:

$$0 = \int_\Omega w \Delta w \, d^n x = \int_\Omega w \nabla \cdot \nabla w \, d^n x = -\int_\Omega |\nabla w|^2 \, d^n x + \int_{\partial\Omega} w \nabla_{\hat{N}} w \, d\sigma. \tag{3.0.8}$$

In the case of Dirichlet data, $w|_{\partial\Omega} = 0$, so the last term in (3.0.8) vanishes. Thus, in the Dirichlet case, we have that

$$\int_\Omega |\nabla w|^2 = 0. \tag{3.0.9}$$

Thus, $\nabla w = 0$ in Ω, and so w is constant in $\overline{\Omega}$. Since w is 0 on $\partial\Omega$, we have that $w \equiv 0$ in $\overline{\Omega}$, which shows that $u \equiv v$ in $\overline{\Omega}$.

Similarly, in the Robin case

$$\int_{\partial\Omega} w \nabla_{\hat{N}} w \, d\sigma = -\alpha \int_{\partial\Omega} w^2 \, d\sigma \le 0, \tag{3.0.10}$$

which implies that

$$\int_\Omega |\nabla w|^2 = 0, \tag{3.0.11}$$

and we can argue as before conclude that $w \equiv 0$ in $\overline{\Omega}$.

Now in the Neumann case, we have that $\nabla_{\hat{N}} w|_{\partial\Omega} = 0$, and we can argue as above to conclude that w is constant in $\overline{\Omega}$. But now we can't say anything about the constant, so the best we can conclude is that $u = v + \text{constant}$ in $\overline{\Omega}$.

□

4. Mean value properties

Harmonic functions u have some amazing properties. Some of the most important ones are captured in the following theorem, which shows that the pointwise values of u can be determined by its average over solid balls or their boundaries.

Theorem 4.1 (Mean value properties). *Let $u(x)$ be harmonic in the domain $\Omega \subset \mathbb{R}^n$, and let $B_R(x) \subset \Omega$ be a ball of radius R centered at the point x. Then the following mean value formulas hold:*

$$u(x) = \frac{n}{\omega_n R^n} \int_{B_R(x)} u(y)\, d^n y, \tag{4.0.12a}$$

$$u(x) = \frac{1}{\omega_n R^{n-1}} \int_{\partial B_R(x)} u(\sigma)\, d\sigma, \tag{4.0.12b}$$

where ω_n is the area of $\partial B_1(0) \subset \mathbb{R}^n$, that is, the area of the boundary of the unit ball in $\mathbb{R}^n$.

Proof. Let's address the $n = 2$ case only; the proof is similar for other values of n. Let's also assume that x is the origin; as we will see, we will be able to treat the case of general x by reducing it to the origin. We will work with polar coordinates (r, θ) on $\mathbb{R}^2$. For a ball of radius r, we have that the measure $d\sigma$ corresponding to $\partial B_r(0)$ is $d\sigma = r\, d\theta$. Note also that along $\partial B_r(0)$, we have that $\partial_r u = \nabla u \cdot \hat{N} = \nabla_{\hat{N}} u$, where $\hat{N}(\sigma)$ is the unit normal to $\partial B_r(0)$. For any $0 \leq r < R$, we define

$$g(r) \overset{\text{def}}{=} \frac{1}{2\pi r} \int_{\partial B_r(0)} u(\sigma)\, d\sigma = \frac{1}{2\pi r} \int_{\theta=0}^{2\pi} r u(r, \theta)\, d\theta = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} u(r, \theta)\, d\theta. \tag{4.0.13}$$

We now note that since u is continuous at 0, we have that

$$u(0) = \lim_{r \to 0^+} g(r). \tag{4.0.14}$$

Thus, we would obtain (4.0.12b) in the case $x = 0$ if we could show that $g'(r) = 0$. Let's now show this. To this end, we calculate that

$$g'(r) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \partial_r u(r, \theta)\, d\theta = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \nabla_{\hat{N}} u(r, \theta)\, d\theta = \frac{1}{2\pi} \int_{\partial B_1(0)} \nabla_{\hat{N}(\sigma)} u(\sigma)\, d\sigma. \tag{4.0.15}$$

By the divergence theorem, this last term is equal to

$$\frac{1}{2\pi} \int_{B_1(0)} \Delta u(y)\, d^2 y. \tag{4.0.16}$$

But $\Delta u = 0$ since u is harmonic, so we have shown that

$$g'(r) = 0, \tag{4.0.17}$$

and we have shown (4.0.12b) for $x = 0$.

To prove (4.0.12a), we use polar coordinate integration and (4.0.12b) (in the case $x = 0$) to obtain

$$u(0)R^2/2 = \int_0^R r u(0)\,dr = \frac{1}{2\pi}\int_0^R \int_{\theta=0}^{2\pi} r u(r,\theta)\,d\theta\,dr = \frac{1}{2\pi}\int_{B_R(0)} u(y)\,d^2y. \tag{4.0.18}$$

We have now shown (4.0.12a) and (4.0.12b) when $x = 0$.

To obtain the corresponding formulas for non-zero x, define $v(y) \stackrel{\text{def}}{=} u(x+y)$, and note that $\Delta_y v(y) = (\Delta_y u)(x+y) = 0$. Therefore, using what we have already shown,

$$u(x) = v(0) = \frac{2}{\omega_n R^2}\int_{B_R(0)} v(y)\,d^2y = \frac{2}{\omega_2 R^2}\int_{B_R(0)} u(x+y)\,d^2y = \frac{2}{\omega_2 R^2}\int_{B_R(x)} u(y)\,d^2y, \tag{4.0.19}$$

which implies (4.0.12a) for general x . We can similarly obtain (4.0.12b) for general x.

□

5. Maximum Principle

Let's now discuss another amazing property verified by harmonic functions. The property, known as the *strong maximum principle*, says that most harmonic functions achieve their maximums and minimums only on the interior of Ω. The only exceptions are the constant functions.

Theorem 5.1 (**Strong Maximum Principle**). *Let $\Omega \subset \mathbb{R}^n$ be a domain, and assume that $u \in C(\Omega)$ verifies the mean value property* (4.0.12a)*. Then if u achieves its max or min at a point $p \in \Omega$, then u is constant on Ω. Therefore, if Ω is bounded and $u \in C(\overline{\Omega})$ is not constant, then for every $x \in \Omega$, we have*

$$u(x) < \max_{y\in\partial\Omega} u(y), \qquad u(x) > \min_{y\in\partial\Omega} u(y). \tag{5.0.20}$$

Proof. We give the argument for the "min" in the case $n = 2$. Suppose that u achieves its min at a point $p \in \Omega$, and that $u(p) = m$. Let $B(p) \subset \Omega$ be any ball centered at p, and let z be any point in $B(p)$. Choose a small ball $B_r(z)$ of radius r centered z with $B_r(z) \subset B(p)$.

Note that by the definition of a min, we have that

$$u(z) \geq m. \tag{5.0.21}$$

Using the assumption that the mean value property (4.0.12a) holds, we conclude that

(5.0.22)

$$\begin{aligned} m &= \frac{1}{|B(p)|}\int_{B(p)} u(y)\,d^2y = \frac{1}{|B(p)|}\left\{\int_{B_r(z)} u(y)\,d^2y + \int_{B\setminus B_r(z)} u(y)\,d^2y\right\} \\ &= \frac{1}{|B(p)|}\left\{|B_r(z)|u(z) + \int_{B\setminus B_r(z)} u(y)\,d^2y\right\} \geq \frac{1}{|B(p)|}\left\{|B_r(z)|u(z) + m\big(|B(p)| - |B_r(z)|\big)\right\}. \end{aligned}$$

Rearranging inequality (5.0.22), we conclude that

$$u(z) \leq m. \tag{5.0.23}$$

Combining (5.0.21) and (5.0.23), we conclude that

$$u(x) = m \tag{5.0.24}$$

holds for *all points* $x \in B(p)$. Therefore, u is locally constant at any point where it achieves its min. Since Ω is open and connected, we conclude that $u(x) = m$ for all $x \in \Omega$.

□

The next corollary will allow us to compare the size of two solutions to Poisson's equation if we have information about the size of the source terms and about the values of the solutions on $\partial\Omega$. The proof is based on Theorem 5.1.

Corollary 5.0.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $f \in C(\Omega)$. Then the PDE*

$$\begin{cases} \Delta u = 0, & x \in \Omega, \\ u(x) = f(x), & x \in \partial\Omega, \end{cases} \tag{5.0.25}$$

has at most one solution $u_f \in C^2(\Omega) \cap C(\overline{\Omega})$. Furthermore, if u_f and u_g are the solutions corresponding to the data $f, g \in C(\overline{\Omega})$, then

1. *(**Comparison Principle**) If $f \geq g$ on $\partial\Omega$ and $f \neq g$, then*
$$u_f > u_g \text{ in } \Omega.$$
2. *(**Stability Estimate**) For any $x \in \Omega$, we have that*
$$|u_f(x) - u_g(x)| \leq \max_{y \in \partial\Omega} |f(y) - g(y)|.$$

Proof. We first prove the Comparison Principle. Let $w = u_f - u_g$. Then by subtracting the PDEs, we see that w solves

$$\begin{cases} \Delta w = 0, & x \in \Omega, \\ u(x) = f(x) - g(x) \geq 0, & x \in \partial\Omega, \end{cases} \tag{5.0.26}$$

Since w is harmonic, since $f(x) - g(x) \geq 0$ on $\partial\Omega$, and since $f \neq g$, Theorem 5.1 implies that w is not constant and that for every $x \in \Omega$, we have

$$w(x) > \max_{y \in \partial\Omega} f(y) - g(y) \geq 0. \tag{5.0.27}$$

This proves the Comparison Principle.

For the Stability Estimate, we perform a similar argument for both $\pm w$, which leads to the estimates

$$w(x) > -\max_{y \in \partial\Omega} |f(y) - g(y)|, \tag{5.0.28}$$

$$-w(x) > -\max_{y \in \partial\Omega} |f(y) - g(y)|. \tag{5.0.29}$$

Combining (5.0.28) and (5.0.29), we deduce the Stability Estimate.

The "at most one" statement of the corollary now follows directly from applying the Stability Estimate to w in the case $f = g$. □

18.152 Introduction to Partial Differential Equations.
Fall 2011