MATH 18.152 COURSE NOTES - CLASS MEETING # 7

18.152 Introduction to PDEs, Fall 2011 Professor: Jared Speck

Class Meeting # 7: The Fundamental Solution and Green Functions

1. The Fundamental Solution for Δ in $\mathbb{R}^n$

Here is a situation that often arises in physics. We are given a function $f(x)$ on $\mathbb{R}^n$ representing the spatial density of some kind of quantity, and we want to solve the following equation:

$$\Delta u(x) = f(x), \qquad x = (x^1, \cdots, x^n) \in \mathbb{R}^n. \tag{1.0.1}$$

Furthermore, we often want to impose the following decay condition as $|x| \to \infty$:

$$|u(x)| \to 0. \tag{1.0.2}$$

For technical reasons, we will need a different condition in the case $n = 2$. A good physical example is the theory of electrostatics, in which $u(x)$ is the electric potential[1], and $f(x)$ is the charge density. $f(x)$ could be e.g a compactly supported function modeling the charge density of a charged star, and we might want to know how the potential behaves far away from the star (i.e. as $|x| \to \infty$). Roughly speaking, the decay conditions (1.0.2) are physically motivated by the fact that the star should not have a large effect on far-away locations.

As we will soon see, the PDE (1.0.1) has a unique solution verifying (1.0.2) as long as $f(x)$ is sufficiently differentiable and decays sufficiently rapidly as $|x| \to \infty$. Much like in the case of the heat equation, we will be able to construct the solution using an object called the *fundamental solution.*

Definition 1.0.1. The fundamental solution Φ corresponding to the operator Δ is

$$\Phi(x) \overset{\text{def}}{=} \begin{cases} \frac{1}{2\pi} \ln|x| & n = 2, \\ -\frac{1}{\omega_n |x|^{n-2}} & n \geq 3, \end{cases} \tag{1.0.3}$$

where as usual $|x| \overset{\text{def}}{=} \sqrt{\sum_{i=1}^n (x^i)^2}$ and ω_n is the surface area of a unit ball in $\mathbb{R}^n$ (e.g. $\omega_3 = 4\pi$).

Remark 1.0.1. Some people prefer to define their Φ to be the negative of our Φ.

Essentially, our goal in this section is to show that $\Delta\Phi(x) = \delta(x)$, where δ is the delta distribution. Let's assume that this holds for now. We then claim that the solution to (1.0.1) is $u(x) = f * \Phi(x) = \int_{\mathbb{R}^n} f(y)\Phi(x-y)\, d^n y$. This can be heuristically justified by the following heuristic computations: $\Delta_x (f * \Phi) = f * \Delta_x \Phi = f * \delta = f(x)$.

Let's now make rigorous sense of this. We first show that away from the origin, the fundamental solution verifies Laplace's equation.

Lemma 1.0.1. *If $x \neq 0$, then $\Delta\Phi(x) = 0$.*

[1] Recall that the the force $\mathbf{F}$ associated to u is $\mathbf{F} = -\nabla u$.

Proof. Let's do the proof in the case $n = 3$. Note that $\Phi(x) = \Phi(r)$ ($r \stackrel{\text{def}}{=} |x|$) is spherically symmetric. Thus, using the fact that $\Delta = \partial_r^2 + \frac{2}{r}\partial_r$ when $r > 0$ for spherically symmetric functions, we have that $\Delta\Phi = \partial_r^2\Phi + \frac{2}{r}\partial_r\Phi = \frac{-2}{\omega_3 r^4} + \frac{2}{\omega_3 r^4} = 0$. □

We are now ready to state and prove a rigorous version of the aforementioned heuristic results.

Theorem 1.1 (**Solution to Poisson's equation in** $\mathbb{R}^n$)**.** *Let $f(x) \in C_0^\infty(\mathbb{R}^n)$ (i.e., $f(x)$ is a smooth, compactly supported function on $\mathbb{R}^n$). Then for $n \geq 3$, the Laplace equation $\Delta u(x) = f(x)$ has a unique smooth solution $u(x)$ that tends to 0 as $|x| \to \infty$. For $n = 2$, the solution is unique under the assumptions $\frac{u(x)}{|x|} \to 0$ as $|x| \to \infty$ and $|\nabla u(x)| \to 0$ as $|x| \to \infty$. Furthermore, these unique solutions can be represented as*

$$u(x) = (\Phi * f)(x) = \begin{cases} \frac{1}{2\pi}\int_{\mathbb{R}^2} \ln|y| f(x-y)\, d^2y, & n = 2, \\ -\frac{1}{\omega_n}\int_{\mathbb{R}^n} |y|^{n-2} f(x-y)\, d^n y, & n \geq 3. \end{cases} \tag{1.0.4}$$

Furthermore, there exist constants $C_n > 0$ such that the following decay estimate holds for the solution as $|x| \to \infty$:

$$|u(x)| \leq \begin{cases} C_2 \ln|x| & n = 2, \\ \frac{C_n}{|x|^{n-2}} & n \geq 3. \end{cases} \tag{1.0.5}$$

Remark 1.0.2. As we alluded to above, Theorem 1.1 shows that $\Delta\Phi(x) = \delta(x)$, where δ is the "delta distribution." For on the one hand, as we have previously discussed, we have that $f = \delta * f$. On the other hand, our proof of Theorem 1.1 below will show that $f = \Delta u = \Delta(\Phi * f) = (\Delta\Phi) * f$. Thus, for any f, we have $\delta * f = (\Delta\Phi) * f$, and so $\Delta\Phi = \delta$.

Proof. We consider only the case $n = 3$. Let's first show existence by checking that the function u defined in (1.0.4) solves the equation and has the desired properties. We first differentiate under the integral (we use one of our prior propositions to justify this) and use the fact that $\Delta_x f(x-y) = \Delta_y f(x-y)$ (you can easily verify this with the chain rule) to derive

$$\Delta_x u(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|y|}\Delta_x f(x-y)\, d^3y = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|y|}\Delta_y f(x-y)\, d^3y. \tag{1.0.6}$$

To show that the right-hand side of (1.0.6) is equal to $f(x)$, we will split the integral into two pieces: a small ball centered at the origin, and it's complement. Thus, let $B_\epsilon(0)$ denote the ball of radius ϵ centered at 0. We then split

$$\Delta_x u(x) = -\frac{1}{4\pi}\int_{B_\epsilon(0)} \frac{1}{|y|}\Delta_y f(x-y)\, d^3y - \frac{1}{4\pi}\int_{B_\epsilon^c(0)} \frac{1}{|y|}\Delta_y f(x-y)\, d^3y \stackrel{\text{def}}{=} I + II. \tag{1.0.7}$$

We first show that I goes to 0 as $\epsilon \to 0^+$. To this end, let

$$M \stackrel{\text{def}}{=} \sup_{y\in\mathbb{R}^3} |f(y)| + |\nabla f(y)| + |\Delta_y f(y)|. \tag{1.0.8}$$

Then using spherical coordinates (r, ω) for the y variable, and recalling that $d^3y = r^2\, d\omega$ (where $\omega \in \partial B_1(0) \subset \mathbb{R}^3$ is a point on the unit sphere and $d\omega = \sin\theta\, d\theta\, d\phi$) in spherical coordinates, we have that

(1.0.9)
$$|I| \le \int_{B_\epsilon(0)} \Big|\frac{1}{|y|}\Delta_y f(x-y)\Big| \, d^3y \le M \int_{r=0}^{\epsilon}\int_{\partial B_1(0)} r\, d\omega\, dr = 2\epsilon^2 \pi M.$$

Clearly, the right-hand side of (1.0.9) goes to 0 as $\epsilon \to 0^+$.

We would now like to understand the second term on the right-hand side of (1.0.7). We claim that

(1.0.10)
$$|f(x) - II| \to 0$$

as $\epsilon \to 0^+$. After we show this, we can combine (1.0.7), (1.0.9), and (1.0.10) and let $\epsilon \to 0^+$ to deduce that $\Delta_x u(x) = f(x)$ as desired.

To show (1.0.10), we will use integration by parts via *Green's identity* and simple estimates to control the boundary terms. Recall that Green's identity for two functions u, v is

(1.0.11)
$$\int_\Omega v(x)\Delta u(x) - u(x)\Delta v(x)\, d^n x = \int_{\partial\Omega} v\nabla_{\hat{N}} u(\sigma) - u\nabla_{\hat{N}} v(\sigma)\, d\sigma.$$

Using (1.0.11) and Lemma 1.0.1, we compute that

(1.0.12)
$$\int_{B_\epsilon^c(0)} -\frac{1}{|y|}\Delta_y f(x-y) + f(x-y) \overbrace{\Delta_y \frac{1}{|y|}}^{0} \, d^3y = \int_{\partial B_\epsilon^c(0)} \frac{1}{|\sigma|}\nabla_{\hat{N}(\sigma)} f(x-\sigma) - f(x-\sigma)\nabla_{\hat{N}(\sigma)}\frac{1}{|\sigma|}\, d\sigma.$$

Above, $\nabla_{\hat{N}(\sigma)}$ is the outward unit radial derivative on the sphere $\partial B_\epsilon(0)$. This corresponds to the "opposite" choice of normal that appears in the standard formulation of Green's identity for $B_\epsilon^c(0)$, but we have compensated for this by carefully inserting minus signs on the right-hand side of (1.0.12). Recalling also that $\nabla_{\hat{N}(\sigma)}\frac{1}{|\sigma|} = -\frac{1}{|\sigma|^2}$, that $|\sigma| = \epsilon$ on $\partial B_\epsilon^c(0)$, and that $d\sigma = \epsilon^2 d\omega$ on $\partial B_\epsilon^c(0)$, it follows that

(1.0.13)
$$-\int_{B_\epsilon^c(0)} \frac{1}{|y|}\Delta_y f(x-y)\, d^3y = -\int_{\partial B_1(0)} \epsilon\omega\cdot(\nabla f)(x-\epsilon\omega)\, d\omega + \int_{\partial B_1(0)} f(x-\epsilon\omega)\, d\omega.$$

Using (1.0.8), it follows that the first integral on the right-hand side of (1.0.13) is bounded by $4\pi M\epsilon$, and thus goes to 0 as $\epsilon \to 0^+$. Furthermore, since f is continuous and since $\int_{\partial B_1(0)} 1\, d\omega = 4\pi$, it follows that the second integral converges to $4\pi f(x)$ as $\epsilon \to 0^+$. We have thus proved (1.0.10) for $n = 3$.

To estimate $|u(x)|$ as $|x| \to \infty$, we assume that $f(x)$ vanishes outside of the ball $B_R(0)$. It suffices to estimate right-hand side of (1.0.4) when $|x| > 2R$. We first note the inequality $\frac{1}{|x-y|} \le \frac{2}{|x|}$, which holds for $|y| \le R$ and $|x| > 2R$. Using this inequality and (1.0.8), we can estimate right-hand side of (1.0.4) by

(1.0.14)
$$|u(x)| = \frac{1}{4\pi}\Big|\int_{B_R(0)} \frac{1}{|x-y|} f(y)\, d^3y\Big| \le \frac{M}{2\pi|x|}\int_{B_R(0)} 1\, d^3y = \frac{2R^3 M}{3|x|},$$

and we have shown (1.0.5) in the case $n = 3$.

To prove uniqueness, we will make use of Corollary 4.0.4, which we will prove later. Now if u, v are two solutions with the assumed decay conditions at ∞, then using the usual strategy, we note that $w \overset{\text{def}}{=} u - v$ is a solution to the Laplace equation

$$\Delta w = 0 \tag{1.0.15}$$

that verifies $|w(x)| \to 0$ as $|x| \to \infty$. In particular, w is a bounded harmonic function on $\mathbb{R}^3$. We will show in Corollary 4.0.4 below that $w(x)$ must be a constant function. Furthermore, the constant must be 0 since $|w(x)| \to 0$ as $|x| \to \infty$.

□

2. Green functions for domains Ω

Our goal in this section is to derive an analog of Theorem 1.1 on the interior of domains $\Omega \subset \mathbb{R}^n$. Specifically, we will study the boundary value Poisson problem

$$\begin{aligned} \Delta u(x) &= f(x), && x \in \Omega \subset \mathbb{R}^n, \\ u(x) &= g(x), && x \in \partial\Omega. \end{aligned} \tag{2.0.16}$$

Theorem 2.1 (**Basic existence theorem**). *Let g be a bounded Lipschitz domain, and let $g \in C(\partial\Omega)$. Then the PDE* (2.0.16) *has a unique solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$.*

Proof. This proof is a bit beyond this course. □

Definition 2.0.2. Let $\Omega \subset \mathbb{R}^n$ be a domain. A *Green function* in Ω is defined to be a function of $(x, y) \in \Omega \times \Omega$ verifying the following conditions for each fixed $x \in \Omega$:

$$\Delta_y G(x, y) = \delta(x), \qquad y \in \Omega \tag{2.0.17}$$

$$G(x, \sigma) = 0, \qquad \sigma \in \partial\Omega. \tag{2.0.18}$$

Proposition 2.0.2. *Let Φ be the fundamental solution* (1.0.3) *for Δ in $\mathbb{R}^n$, and let $\Omega \in \mathbb{R}^n$ be a domain. Then the Green function $G(x, y)$ for Ω can be decomposed as*

$$G(x, y) = \Phi(x - y) - \phi(x, y), \tag{2.0.19}$$

where for each $x \in \Omega$, $\phi(x, y)$ solves the Dirichlet problem

$$\Delta_y \phi(x, y) = 0, \qquad y \in \Omega, \tag{2.0.20}$$

$$\phi(x, \sigma) = \Phi(x - \sigma), \qquad \sigma \in \partial\Omega. \tag{2.0.21}$$

Proof. As we have previously discussed, $\Delta\Phi = \delta$. Also using (2.0.20), we compute that

$$\Delta_y\big(\Phi(x - y) - \phi(x, y)\big) = \Delta_y \Phi(x - y) - \Delta_y \phi(x, y) = \delta(x - y). \tag{2.0.22}$$

Therefore, $\Phi(x - y) - \phi(x, y)$ verifies equation (2.0.17).

Furthermore, using (2.0.21), we have that $\Phi(x - \sigma) - \phi(x, \sigma) = 0$ whenever $\sigma \in \partial\Omega$. Thus, $\Phi(x - y) - \phi(x, y)$ also verifies the boundary condition (2.0.18).

□

The following technical proposition will play later in this section when we derive representation formulas for solutions to (2.0.16) in terms of Green functions.

Proposition 2.0.3 (Representation formula for u). *Let Φ be the fundamental solution* (1.0.3) *for Δ in $\mathbb{R}^n$, and let $\Omega \subset \mathbb{R}^n$ be a domain. Assume that $u \in C^2(\overline{\Omega})$. Then for every $x \in \Omega$, we have the following representation formula for $u(x)$:*

$$u(x) = \int_{\Omega} \Phi(x-y)\Delta_y u(y)\, d^n y - \underbrace{\int_{\partial\Omega} \Phi(x-\sigma)\nabla_{\hat{N}(\sigma)} u(\sigma)\, d\sigma}_{\textit{single layer potential}} + \underbrace{\int_{\partial\Omega} u(\sigma)\nabla_{\hat{N}(\sigma)}\Phi(x-\sigma)\, d\sigma}_{\textit{double layer potential}}. \tag{2.0.23}$$

Proof. We'll do the proof for $n = 3$, in which case $\Phi(x) = -\frac{1}{4\pi|x|}$. We will also make use of Green's identity (1.0.11). Let $B_\epsilon(x)$ be a ball of radius ϵ centered at x, and let $\Omega_\epsilon \overset{\text{def}}{=} \Omega\backslash B_\epsilon(x)$. Note that $\partial\Omega_\epsilon = \partial\Omega \cup -\partial B_\epsilon(x)$. Using (1.0.11), we compute that

$$\begin{aligned} \int_{\Omega_\epsilon} \frac{1}{|x-y|}\Delta u(y)\, d^3y &= \int_{\partial\Omega_\epsilon} \frac{1}{|x-\sigma|}\nabla_{\hat{N}} u(\sigma) - u(\sigma)\nabla_{\hat{N}}\Big(\frac{1}{|x-\sigma|}\Big)\, d\sigma \\ &= \int_{\partial\Omega} \frac{1}{|x-\sigma|}\nabla_{\hat{N}} u(\sigma)\, d\sigma - \int_{\partial\Omega} u(\sigma)\nabla_{\hat{N}}\Big(\frac{1}{|x-\sigma|}\Big)\, d\sigma \\ &\quad - \int_{\partial B_\epsilon(x)} \frac{1}{|x-\sigma|}\nabla_{\hat{N}} u(\sigma)\, d\sigma + \int_{\partial B_\epsilon(x)} u(\sigma)\nabla_{\hat{N}}\Big(\frac{1}{|x-\sigma|}\Big)\, d\sigma. \end{aligned} \tag{2.0.24}$$

In the last two integrals above, $\hat{N}(\sigma)$ denotes the radially outward unit normal to the boundary of the ball $B_\epsilon(x)$. This corresponds to the "opposite" choice of normal that appears in the standard formulation of Green's identity, but we have compensated by adjusting the signs in front of the integrals.

Let's symbolically write (2.0.24) as

$$L = R1 + R2 + R3 + R4. \tag{2.0.25}$$

Our goal is to show that as $\epsilon \downarrow 0$, the following limits are achieved:

- $L \to -4\pi \int_\Omega \Phi(x-y)\Delta_y u(y)\, d^3y$
- $R1 \to 4\pi \times$ single layer potential
- $R2 \to -4\pi \times$ double layer potential
- $R3 \to 0$
- $R4 \to -4\pi u(x)$.

Once we have calculated the above limits, (2.0.23) then follows from simple algebraic rearranging.

We first address L. Let $M = \max_{y\in\overline{\Omega}} \Delta u(y)$. We then estimate

$$\begin{aligned} \Big| \int_{\Omega} \frac{1}{|x-y|}\Delta u(y)\, d^3y - \int_{\Omega_\epsilon} \frac{1}{|x-y|}\Delta u(y)\, d^3y \Big| &\le \int_{B_\epsilon(x)} \frac{1}{|x-y|}|\Delta u(y)|\, d^3y \\ &\le M \int_{B_\epsilon(x)} \frac{1}{|x-y|}\, d^3y \to 0 \text{ as } \epsilon \downarrow 0. \end{aligned} \tag{2.0.26}$$

This shows that L converges to $\int_\Omega \frac{1}{|x-y|}\Delta u(y)\, d^3y$ as $\epsilon \downarrow 0$.

The limits for $R1$ and $R2$ are obvious since these terms do not depend on ϵ.

We now address $R3$. To this end, Let $M' = \max_{y\in\overline{\Omega}} |\nabla u(y)|$. We then estimate $R3$ by

(2.0.27)

$$|R3| \leq \int_{\partial B_\epsilon(x)} \left| \frac{1}{|x-\sigma|} \nabla_{\hat{N}} u(\sigma) \right| d\sigma \leq \int_{\partial B_\epsilon(x)} \frac{1}{\epsilon} M' \, d\sigma = \underbrace{4\pi\epsilon^2}_{\text{surface area of } \partial B_\epsilon(x)} \times \epsilon^{-1} M' \to 0 \text{ as } \epsilon \downarrow 0.$$

We now address $R4$. Using spherical coordinates $(r, \theta, \phi) \in [0, \infty) \times [0, \pi) \times [0, 2\pi)$ centered at x, we have that $d\sigma = r^2 \sin\theta \, d\theta \, d\phi$. Therefore, $\int_{\partial B_\epsilon(x)} \frac{1}{|x-\sigma|^2} \, d\sigma = \int_{\phi \in [0,2\pi]} \int_{\theta \in [0,\pi]} 1 \, d\theta \, d\phi = 4\pi$. We now estimate

$$\begin{aligned}
(2.0.28) \qquad \left| \frac{1}{4\pi} R4 - \big[-u(x)\big] \right| &= \left| u(x) + \frac{1}{4\pi} \int_{\partial B_\epsilon(x)} u(\sigma) \nabla_{\hat{N}(\sigma)} \Big(\frac{1}{|x-\sigma|} \Big) \, d\sigma \right| \\
&= \frac{1}{4\pi} \left| \int_{\partial B_\epsilon(x)} \big(u(x) - u(\sigma)\big) \Big(\frac{1}{|x-\sigma|^2} \Big) \, d\sigma \right| \\
&\leq \frac{1}{4\pi} \int_{\partial B_\epsilon(x)} |u(x) - u(\sigma)| \Big(\frac{1}{|x-\sigma|^2} \Big) \, d\sigma \\
&\leq \frac{1}{4\pi} \max_{\sigma \in \partial B_\epsilon(x)} |u(x) - u(\sigma)| \int_{\partial B_\epsilon(x)} \Big(\frac{1}{|x-\sigma|^2} \Big) \, d\sigma \\
&\leq \max_{\sigma \in \partial B_\epsilon(x)} |u(x) - u(\sigma)| \to 0 \text{ as } \epsilon \downarrow 0.
\end{aligned}$$

This shows that $R4 \to -4\pi u(x)$ as $\epsilon \downarrow 0$.

□

Theorem 2.2 (**Representation formula for solutions to the boundary value Poisson equation**). *The solution u to* (2.0.16) *can be represented as*

$$(2.0.29) \qquad u(x) = -\int_\Omega f(y) G(x,y) \, d^n y - \int_{\partial\Omega} g(\sigma) \underbrace{\nabla_{\hat{N}} G(x,\sigma)}_{\textit{Poisson kernel}} \, d\sigma.$$

Proof. Applying Proposition 2.0.3, we have that

$$(2.0.30) \qquad u(x) = -\int_\Omega \Phi(x-y) f(y) \, d^n y + \int_{\partial\Omega} \Phi(x-\sigma) \nabla_{\hat{N}(\sigma)} u(\sigma) \, d\sigma - \int_{\partial\Omega} g(\sigma) \nabla_{\hat{N}(\sigma)} \Phi(x-\sigma) \, d\sigma.$$

Recall also that

$$(2.0.31) \qquad G(x,y) = \Phi(x-y) - \phi(x,y)$$

$$(2.0.32) \qquad G(x,\sigma) = 0 \text{ when } \sigma \in \partial\Omega.$$

Applying the Green identity (1.0.11) to the functions $u(y)$ and $\phi(x,y)$, and recalling that $\Delta_y \phi(x,y) = 0$ for each fixed $x \in \Omega$, we have that

$$(2.0.33) \qquad 0 = \int_\Omega \phi(x,y) \overbrace{f(y)}^{\Delta u(y)} \, d^n y - \int_{\partial\Omega} \overbrace{\phi(x,\sigma)}^{\Phi(x-\sigma)} \nabla_{\hat{N}} u(\sigma) \, d\sigma + \int_{\partial\Omega} \overbrace{g(\sigma)}^{u(\sigma)} \nabla_{\hat{N}} \phi(x,\sigma) \, d\sigma.$$

Adding (2.0.30) and (2.0.33), and using (2.0.31), we deduce the formula (2.0.29).

□

3. Poisson's Formula

Let's compute the Green function $G(x,y)$ and Poisson kernel $P(x,\sigma) \overset{\text{def}}{=} -\nabla_{\hat{N}} G(x,\sigma)$ from (2.0.29) in the case that $\Omega \overset{\text{def}}{=} B_R(0) \subset \mathbb{R}^3$ is a ball of radius R centered at the origin. We'll use a technique called the *method of images* that works for special domains.

Warning 3.0.1. Brace yourself for a bunch of tedious computations that at the end of the day will lead to a very nice expression.

The basic idea is to hope that $\phi(x,y)$ from (2.0.19), viewed as a potential that depends on y, is equal to the potential generated by some "imaginary charge" q placed at a point $x^* \in \Omega^c$. To ensure that property (2.0.18) holds, q and x^* have to be chosen so that along the boundary $\{y \in \mathbb{R}^3 \mid |y| = R\}$, $\phi(x,y) = \frac{1}{4\pi|x-y|}$. In a nutshell, we guess that

$$G(x,y) = \frac{1}{4\pi|x-y|} - \frac{q}{4\pi|x^*-y|}, \tag{3.0.34}$$

and we try to solve for q and x^* so that $G(x,y)$ vanishes when $|y| = R$. Thus, when $|y| = R$, we must have

$$\frac{1}{4\pi|x-y|} = \frac{q}{4\pi|x^*-y|}. \tag{3.0.35}$$

$$|x^*-y|^2 = q^2|x-y|^2. \tag{3.0.36}$$

$$|x|^2 - 2x\cdot y + R^2 = |x-y|^2 = q^2|x^*-y|^2 = q^2(|x^*|^2 - 2x^*\cdot y + R^2). \tag{3.0.37}$$

Then performing simple algebra, we have

$$|x^*|^2 + R^2 - q^2(R^2 + |x|^2) = 2y\cdot(x^* - q^2 x). \tag{3.0.38}$$

Now since the left-hand side of (3.0.38) does not depend on y, it must be the case that the second term on the right-hand side vanishes. This implies that $x^* = q^2 x$, and also leads to the equation

$$q^4|x|^2 - q^2(R^2 + |x|^2) + R^2 = 0. \tag{3.0.39}$$

Solving (3.0.39) for q, we finally have that

$$q = \frac{R}{|x|}, \tag{3.0.40}$$

$$x^* = \frac{R^2}{|x|^2}x. \tag{3.0.41}$$

Therefore,

$$\phi(x,y) = \frac{1}{4\pi}\frac{R}{|x|\left|\frac{R^2}{|x|^2}x - y\right|}, \tag{3.0.42}$$

$$\phi(0,y) = \frac{1}{R}, \tag{3.0.43}$$

where we took a limit as $x \to 0$ in (3.0.42) to derive (3.0.43).

Next, using (3.0.34), we have

$$G(x,y) = \frac{1}{4\pi|x-y|} - \frac{1}{4\pi}\frac{R}{|x|\left|\frac{R^2}{|x|^2}x - y\right|}, \qquad x \neq 0, \tag{3.0.44}$$

$$G(0,y) = \frac{1}{4\pi|y|} - \frac{1}{R}. \tag{3.0.45}$$

$$\nabla_y G(x,y) = \frac{x-y}{4\pi|x-y|^3} - \frac{1}{4\pi}\frac{R}{|x|}\frac{x^* - y}{|x^* - y|^3} \tag{3.0.46}$$

Now when $\sigma \in \partial B_R(0)$, (3.0.36) and (3.0.40) imply that

$$|x^* - \sigma| = \frac{R}{|x|}|x - \sigma|. \tag{3.0.47}$$

Therefore, using (3.0.46) and (3.0.47), we compute that

$$\begin{aligned}\nabla_\sigma G(x,\sigma) &= \frac{x-\sigma}{4\pi|x-\sigma|^3} - \frac{1}{4\pi}\frac{|x|^2}{R^2}\frac{x^* - \sigma}{|x-\sigma|^3} = \frac{x-\sigma}{4\pi|x-\sigma|^3} - \frac{1}{4\pi}\frac{|x|^2}{R^2}\frac{\frac{R^2}{|x|^2}x - \sigma}{|x-\sigma|^3} \\ &= \frac{-\sigma}{4\pi|x-\sigma|^3}\Big(1 - \frac{|x|^2}{R^2}\Big).\end{aligned} \tag{3.0.48}$$

Using (3.0.48) and the fact that $\hat{N}(\sigma) = \frac{\sigma}{R}$, we deduce

$$\nabla_{\hat{N}(\sigma)} G(x,\sigma) \overset{\text{def}}{=} \nabla_\sigma G(x,\sigma) \cdot \hat{N}(\sigma) = \frac{R^2 - |x|^2}{4\pi R}\frac{1}{|x-\sigma|^3}. \tag{3.0.49}$$

Remark 3.0.3. If the ball were centered at the point $p \in \mathbb{R}^3$ instead of the origin, then the formula (3.0.49) would be replaced with

$$\nabla_{\hat{N}(\sigma)} G(x,\sigma) \overset{\text{def}}{=} \nabla_\sigma G(x,\sigma) \cdot \hat{N}(\sigma) = \frac{R^2 - |x-p|^2}{4\pi R}\frac{1}{|x-\sigma|^3}. \tag{3.0.50}$$

Theorem 3.1 (**Poisson's formula**). *Let $B_R(p) \subset \mathbb{R}^3$ be a ball of radius R centered at $p = (p^1, p^2, p^3)$, and let $x = (x^1, x^2, x^3)$ denote a point in $\mathbb{R}^3$. Then the unique solution $u \in C^2(B_R(p)) \cap C(\overline{B}_R(p))$ of the PDE*

$$\begin{cases} \Delta u = 0, & x \in \Omega, \\ u(x) = f(x), & x \in \partial\Omega, \end{cases} \tag{3.0.51}$$

can be represented using the Poisson formula:

$$u(x) = \frac{R^2 - |x-p|^2}{4\pi R} \int_{\partial B_R(p)} \frac{f(\sigma)}{|x-\sigma|^3} \, d\sigma. \tag{3.0.52}$$

Remark 3.0.4. In n dimensions, the formula (3.0.52) gets replaced with

$$u(x) = \frac{R^2 - |x-p|^2}{\omega_n R} \int_{\partial B_R(p)} \frac{f(\sigma)}{|x-\sigma|^n} \, d\sigma, \tag{3.0.53}$$

where as usual, ω_n is the surface area of the unit ball in $\mathbb{R}^n$.

Proof. The identity (3.0.52) follows immediately from Theorem 2.2 and (3.0.50). □

4. Harnack's inequality

Theorem 4.1 (Harnack's inequality). *Let u be harmonic and* ***non-negative*** *in the ball $B_R(0) \subset \mathbb{R}^n$. Then for any $x \in B_R(0)$, we have that*

$$\frac{R^{n-2}(R-|x|)}{(R+|x|)^{n-1}} u(0) \leq u(x) \leq \frac{R^{n-2}(R+|x|)}{(R-|x|)^{n-1}} u(0). \tag{4.0.54}$$

Proof. We'll do the proof for $n = 3$. The basic idea is to combine the Poisson representation formula with simple inequalities and the mean value property. By Theorem 3.1, we have that

$$u(x) = \frac{R^2 - |x|^2}{4\pi R} \int_{\partial B_R(0)} \frac{f(\sigma)}{|x-\sigma|^3} \, d\sigma. \tag{4.0.55}$$

By the triangle inequality, for $\sigma \in \partial B_R(0)$ (i.e. $|\sigma| = R$), we have that $|x| - R \leq |x-\sigma| \leq |x| + R$. Applying the first inequality to (4.0.55), and **using the non-negativity of** f, we deduce that

$$u(x) \leq \frac{R+|x|}{R^2-|x|^2} \frac{1}{4\pi R} \int_{\partial B_R(0)} f(\sigma) \, d\sigma. \tag{4.0.56}$$

Now recall that by the mean value property, we have that

$$u(0) = \frac{1}{4\pi R^2} \int_{\partial B_R(0)} f(\sigma) \, d\sigma. \tag{4.0.57}$$

Thus, combining (4.0.56) and (4.0.57), we have that

$$u(x) \leq \frac{R^{n-2}(R+|x|)}{(R-|x|)^{n-1}}, \tag{4.0.58}$$

which implies one of the inequalities in (4.0.54). The other one can be proven similarly using the remaining triangle inequality.

□

Corollary 4.0.4 (Liouville's theorem). *Suppose that $u \in C^2(\mathbb{R}^n)$ is harmonic on $\mathbb{R}^n$. Suppose their exists a constant M such that $u(x) \geq M$ for all $x \in \mathbb{R}^n$, or such that $u(x) \leq M$ for all $x \in \mathbb{R}^n$. Then u is constant.*

Proof. We first consider the case that $u(x) \geq M$. Let $v \overset{\text{def}}{=} u + |M|$. Observe that $v \geq 0$ is harmonic and verifies the hypotheses of Theorem 4.1. Thus, by (4.0.54), if $x \in \mathbb{R}^n$ and R is sufficiently large, we have that

$$\frac{R^{n-2}(R - |x|)}{(R + |x|)^{n-1}} u(0) \leq u(x) \leq \frac{R^{n-2}(R + |x|)}{(R - |x|)^{n-1}} u(0). \tag{4.0.59}$$

Allowing $R \to \infty$ in (4.0.59), we conclude that $v(x) = v(0)$. Thus, v is a constant-valued function (and therefore u is too).

To handle the case $u(x) \leq M$, we simply consider the function $w(x) \overset{\text{def}}{=} -u(x) + |M|$ in place of $v(x)$, and we argue as above.

□

18.152 Introduction to Partial Differential Equations.
Fall 2011