

MATH 18.152 COURSE NOTES - CLASS MEETING # 8

18.152 Introduction to PDEs, Fall 2011

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Class Meeting # 8: Green Functions

1. GREEN FUNCTIONS FOR DOMAINS Ω

Our goal in this section is to derive an integral representation formula for the solution to Poisson's equation on domains $\Omega \subset \mathbb{R}^n$. Specifically, we will study the boundary value Poisson PDE

$$(1.0.1) \quad \begin{aligned} \Delta u(x) &= f(x), & x \in \Omega \subset \mathbb{R}^n, \\ u(x) &= g(x), & x \in \partial\Omega. \end{aligned}$$

We first state a basic existence theorem.

Theorem 1.1 (Basic existence theorem). *Let Ω be a bounded Lipschitz domain, and let $g \in C(\partial\Omega)$. Then the PDE (1.0.1) has a unique solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$.*

Proof. This proof is a bit beyond this course. □

We now define the basic object that will play the role of a fundamental solution on a domain Ω .

Definition 1.0.1. Let $\Omega \subset \mathbb{R}^n$ be a domain. A *Green function* in Ω is defined to be a function of $(x, y) \in \Omega \times \Omega$ verifying the following conditions for each fixed $x \in \Omega$:

$$(1.0.2) \quad \Delta_y G(x, y) = \delta_x(y) \stackrel{\text{def}}{=} \delta(y - x), \quad y \in \Omega,$$

$$(1.0.3) \quad G(x, \sigma) = 0, \quad \sigma \in \partial\Omega.$$

Let's now connect $G(x, y)$ to $\Phi(x - y)$.

Proposition 1.0.1. *Let Φ be the fundamental solution for Δ in \mathbb{R}^n , and let $\Omega \in \mathbb{R}^n$ be a domain. Then the Green function $G(x, y)$ for Ω can be decomposed as*

$$(1.0.4) \quad G(x, y) = \Phi(x - y) - \phi(x, y),$$

where for each $x \in \Omega$, $\phi(x, y)$ solves the Dirichlet problem

$$(1.0.5) \quad \Delta_y \phi(x, y) = 0, \quad y \in \Omega,$$

$$(1.0.6) \quad \phi(x, \sigma) = \Phi(x - \sigma), \quad \sigma \in \partial\Omega.$$

Proof. As we have previously discussed, $\Delta\Phi = \delta$. Also using (1.0.5), we compute that

$$(1.0.7) \quad \Delta_y(\Phi(x - y) - \phi(x, y)) = \Delta_y\Phi(x - y) - \Delta_y\phi(x, y) = \delta(y - x).$$

Therefore, $\Phi(x - y) - \phi(x, y)$ verifies equation (1.0.2).

Furthermore, using (1.0.6), we have that $\Phi(x - \sigma) - \phi(x, \sigma) = 0$ whenever $\sigma \in \partial\Omega$. Thus, $\Phi(x - y) - \phi(x, y)$ also verifies the boundary condition (1.0.3). \square

The following technical proposition will play later in this section when we derive representation formulas for solutions to (1.0.1) in terms of Green functions.

Proposition 1.0.2 (Representation formula for u). *Let Φ be the fundamental solution for Δ in \mathbb{R}^n , and let $\Omega \subset \mathbb{R}^n$ be a domain. Assume that $u \in C^2(\bar{\Omega})$. Then for every $x \in \Omega$, we have the following representation formula for $u(x)$:*

$$(1.0.8) \quad u(x) = \int_{\Omega} \Phi(x - y) \Delta_y u(y) d^n y - \underbrace{\int_{\partial\Omega} \Phi(x - \sigma) \nabla_{\hat{N}(\sigma)} u(\sigma) d\sigma}_{\text{single layer potential}} + \underbrace{\int_{\partial\Omega} u(\sigma) \nabla_{\hat{N}(\sigma)} \Phi(x - \sigma) d\sigma}_{\text{double layer potential}}.$$

Proof. We'll do the proof for $n = 3$, in which case $\Phi(x) = -\frac{1}{4\pi|x|}$. We will also make use of Green's identity. Let $B_\epsilon(x)$ be a ball of radius ϵ centered at x , and let $\Omega_\epsilon \stackrel{\text{def}}{=} \Omega \setminus B_\epsilon(x)$. Note that $\partial\Omega_\epsilon = \partial\Omega \cup -\partial B_\epsilon(x)$. Using Green's identity, we compute that

$$(1.0.9) \quad \begin{aligned} \int_{\Omega_\epsilon} \frac{1}{|x - y|} \Delta u(y) d^3 y &= \int_{\partial\Omega_\epsilon} \frac{1}{|x - \sigma|} \nabla_{\hat{N}} u(\sigma) - u(\sigma) \nabla_{\hat{N}} \left(\frac{1}{|x - \sigma|} \right) d\sigma \\ &= \int_{\partial\Omega} \frac{1}{|x - \sigma|} \nabla_{\hat{N}} u(\sigma) d\sigma - \int_{\partial\Omega} u(\sigma) \nabla_{\hat{N}} \left(\frac{1}{|x - \sigma|} \right) d\sigma \\ &\quad - \int_{\partial B_\epsilon(x)} \frac{1}{|x - \sigma|} \nabla_{\hat{N}} u(\sigma) d\sigma + \int_{\partial B_\epsilon(x)} u(\sigma) \nabla_{\hat{N}} \left(\frac{1}{|x - \sigma|} \right) d\sigma. \end{aligned}$$

In the last two integrals above, $\hat{N}(\sigma)$ denotes the radially outward unit normal to the boundary of the ball $B_\epsilon(x)$. This corresponds to the “opposite” choice of normal that appears in the standard formulation of Green's identity, but we have compensated by adjusting the signs in front of the integrals.

Let's symbolically write (1.0.9) as

$$(1.0.10) \quad L = R1 + R2 + R3 + R4.$$

Our goal is to show that as $\epsilon \downarrow 0$, the following limits are achieved:

- $L \rightarrow -4\pi \int_{\Omega} \Phi(x - y) \Delta_y u(y) d^3 y$
- $R1 \rightarrow 4\pi \times \text{single layer potential}$
- $R2 \rightarrow -4\pi \times \text{double layer potential}$
- $R3 \rightarrow 0$
- $R4 \rightarrow -4\pi u(x)$.

Once we have calculated the above limits, (1.0.8) then follows from simple algebraic rearranging.

We first address L . Let $M = \max_{y \in \bar{\Omega}} |\Delta u(y)|$. We then estimate

$$(1.0.11) \quad \begin{aligned} \left| \int_{\Omega} \frac{1}{|x - y|} \Delta u(y) d^3 y - \int_{\Omega_\epsilon} \frac{1}{|x - y|} \Delta u(y) d^3 y \right| &\leq \int_{B_\epsilon(x)} \frac{1}{|x - y|} |\Delta u(y)| d^3 y \\ &\leq M \int_{B_\epsilon(x)} \frac{1}{|x - y|} d^3 y \rightarrow 0 \text{ as } \epsilon \downarrow 0. \end{aligned}$$

This shows that L converges to $\int_{\Omega} \frac{1}{|x-y|} \Delta u(y) d^3y$ as $\epsilon \downarrow 0$.

The limits for $R1$ and $R2$ are obvious since these terms do not depend on ϵ .

We now address $R3$. To this end, Let $M' = \max_{y \in \bar{\Omega}} |\nabla u(y)|$. We then estimate $R3$ by

(1.0.12)

$$|R3| \leq \int_{\partial B_{\epsilon}(x)} \left| \frac{1}{|x-\sigma|} \nabla_{\hat{N}} u(\sigma) \right| d\sigma \leq \int_{\partial B_{\epsilon}(x)} \frac{1}{\epsilon} M' d\sigma = \underbrace{4\pi\epsilon^2}_{\text{surface area of } \partial B_{\epsilon}(x)} \times \epsilon^{-1} M' \rightarrow 0 \text{ as } \epsilon \downarrow 0.$$

We now address $R4$. Using spherical coordinates $(r, \theta, \phi) \in [0, \infty) \times [0, \pi) \times [0, 2\pi)$ centered at x , we have that $d\sigma = r^2 \sin \theta d\theta d\phi$. Therefore, $\int_{\partial B_{\epsilon}(x)} \frac{1}{|x-\sigma|^2} d\sigma = \int_{\phi \in [0, 2\pi]} \int_{\theta \in [0, \pi]} 1 d\theta d\phi = 4\pi$. We now estimate

$$\begin{aligned} (1.0.13) \quad \left| \frac{1}{4\pi} R4 - [-u(x)] \right| &= \left| u(x) + \frac{1}{4\pi} \int_{\partial B_{\epsilon}(x)} u(\sigma) \nabla_{\hat{N}(\sigma)} \left(\frac{1}{|x-\sigma|} \right) d\sigma \right| \\ &= \frac{1}{4\pi} \left| \int_{\partial B_{\epsilon}(x)} (u(x) - u(\sigma)) \left(\frac{1}{|x-\sigma|^2} \right) d\sigma \right| \\ &\leq \frac{1}{4\pi} \int_{\partial B_{\epsilon}(x)} |u(x) - u(\sigma)| \left(\frac{1}{|x-\sigma|^2} \right) d\sigma \\ &\leq \frac{1}{4\pi} \max_{\sigma \in \partial B_{\epsilon}(x)} |u(x) - u(\sigma)| \int_{\partial B_{\epsilon}(x)} \left(\frac{1}{|x-\sigma|^2} \right) d\sigma \\ &\leq \max_{\sigma \in \partial B_{\epsilon}(x)} |u(x) - u(\sigma)| \rightarrow 0 \text{ as } \epsilon \downarrow 0. \end{aligned}$$

This shows that $R4 \rightarrow -4\pi u(x)$ as $\epsilon \downarrow 0$.

□

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