MATH 18.152 COURSE NOTES - CLASS MEETING # 9

18.152 Introduction to PDEs, Fall 2011	Professor: Jared Speck

Class Meeting # 9: Poisson's Formula, Harnack's Inequality, and Liouville's Theorem

1. Representation Formula for Solutions to Poisson's Equation

We now derive our main representation formula for solution's to Poisson's equation on a domain Ω.

Theorem 1.1 (Representation formula for solutions to the boundary value Poisson equation). *Let Ω be a domain with a smooth boundary, and assume that $f \in C^2(\overline{\Omega})$ and $g \in C(\partial\Omega)$. Then the unique solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ to*

$$\Delta u(x) = f(x), \qquad x \in \Omega \subset \mathbb{R}^n, \tag{1.0.1}$$
$$u(x) = g(x), \qquad x \in \partial\Omega.$$

can be represented as

$$u(x) = \int_{\Omega} f(y) G(x,y)\, d^n y + \int_{\partial\Omega} g(\sigma) \underbrace{\nabla_{\hat{N}(\sigma)} G(x,\sigma)}_{\text{Poisson kernel}}\, d\sigma, \tag{1.0.2}$$

where $G(x,y)$ is the Green function for Ω.

Proof. Applying the **Representation formula for** u Proposition, we have that

$$u(x) = \int_{\Omega} \Phi(x-y) f(y)\, d^n y - \int_{\partial\Omega} \Phi(x-\sigma) \nabla_{\hat{N}(\sigma)} u(\sigma)\, d\sigma + \int_{\partial\Omega} g(\sigma) \nabla_{\hat{N}(\sigma)} \Phi(x-\sigma)\, d\sigma. \tag{1.0.3}$$

Recall also that

$$G(x,y) = \Phi(x-y) - \phi(x,y), \tag{1.0.4}$$

where

$$\Delta_y \phi(x,y) = 0, \qquad x \in \Omega, \tag{1.0.5}$$

and

$$G(x,\sigma) = 0 \text{ when } x \in \Omega \text{ and } \sigma \in \partial\Omega. \tag{1.0.6}$$

The expression (1.0.3) is not very useful since don't know the value of $\nabla_{\hat{N}(\sigma)} u(\sigma)$ along $\partial\Omega$. To fix this, we will use Green's identity. Applying Green's identity to the functions $u(y)$ and $\phi(x,y)$, and recalling that $\Delta_y \phi(x,y) = 0$ for each fixed $x \in \Omega$, we have that

(1.0.7)
$$0 = \int_{\Omega} \phi(x,y) \overbrace{f(y)}^{\Delta u(y)} \, d^n y - \int_{\partial\Omega} \overbrace{\phi(x,\sigma)}^{\Phi(x-\sigma)} \nabla_{\hat{N}} u(\sigma) \, d\sigma + \int_{\partial\Omega} \overbrace{g(\sigma)}^{u(\sigma)} \nabla_{\hat{N}} \phi(x,\sigma) \, d\sigma.$$

Subtracting (1.0.7) from (1.0.3), and using (1.0.4), we deduce the formula (1.0.2).

□

2. Poisson's Formula

Let's compute the Green function $G(x,y)$ and Poisson kernel $P(x,\sigma) \overset{\text{def}}{=} \nabla_{\hat{N}} G(x,\sigma)$ from (1.0.2) in the case that $\Omega \overset{\text{def}}{=} B_R(0) \subset \mathbb{R}^3$ is a ball of radius R centered at the origin. We'll use a technique called the *method of images* that works for special domains.

Warning 2.0.1. Brace yourself for a bunch of tedious computations that at the end of the day will lead to a very nice expression.

The basic idea is to hope that $\phi(x,y)$ from the decomposition $G(x,y) = \Phi(x-y) - \phi(x,y)$, where $\phi(x,y)$ is viewed as a function of x that depends on the parameter y, is equal to the Newtonian potential generated by some "imaginary charge" q placed at a point $x^* \in B_R^c(0)$. To ensure that $G(x,\sigma) = 0$ when $\sigma \in \partial B_R(0)$, q and x^* have to be chosen so that along the boundary $\{y \in \mathbb{R}^3 \mid |y| = R\}$, $\phi(x,y) = \frac{1}{4\pi|x-y|}$. In a nutshell, we guess that

(2.0.8)
$$G(x,y) = -\frac{1}{4\pi|x-y|} + \underbrace{\frac{q}{4\pi|x^*-y|}}_{\phi(x,y)?},$$

and we try to solve for q and x^* so that $G(x,y)$ vanishes when $|y| = R$.

Remark 2.0.1. Note that $\Delta_y \frac{q}{4\pi|x^*-y|} = 0$, which is one of the conditions necessary for constructing $G(x,y)$.

By the definition of $G(x,y)$, we must have $G(x,y) = 0$ when $|y| = R$, which implies that

(2.0.9)
$$\frac{1}{4\pi|x-y|} = \frac{q}{4\pi|x^*-y|}.$$

Simple algebra then leads to

(2.0.10)
$$|x^* - y|^2 = q^2|x-y|^2.$$

When $|y| = R$, we use (2.0.10) to compute that

(2.0.11)
$$|x^*|^2 - 2x^* \cdot y + R^2 = |x^* - y|^2 = q^2|x-y|^2 = q^2\big(|x|^2 - 2x \cdot y + R^2\big),$$

where $\cdot$ denotes the Euclidean dot product. Then performing simple algebra, it follows from (2.0.11) that

(2.0.12)
$$|x^*|^2 + R^2 - q^2(R^2 + |x|^2) = 2y \cdot (x^* - q^2 x).$$

Now since the left-hand side of (2.0.12) does not depend on y, it must be the case that the right-hand side is always 0. This implies that $x^* = q^2 x$, and also leads to the equation

$$q^4|x|^2 - q^2(R^2 + |x|^2) + R^2 = 0. \tag{2.0.13}$$

Solving (2.0.13) for q, we finally have that

$$q = \frac{R}{|x|}, \tag{2.0.14}$$

$$x^* = \frac{R^2}{|x|^2}x. \tag{2.0.15}$$

Therefore,

$$\phi(x,y) = \frac{1}{4\pi}\frac{R}{|x|\left|\frac{R^2}{|x|^2}x - y\right|}, \tag{2.0.16}$$

$$\phi(0,y) = \frac{1}{4\pi R}, \tag{2.0.17}$$

where we took a limit as $x \to 0$ in (2.0.16) to derive (2.0.17).

Next, using (2.0.8), we have

$$G(x,y) = -\frac{1}{4\pi|x-y|} + \frac{1}{4\pi}\frac{R}{|x|\left|\frac{R^2}{|x|^2}x - y\right|}, \qquad x \neq 0, \tag{2.0.18}$$

$$G(0,y) = -\frac{1}{4\pi|y|} + \frac{1}{4\pi R}. \tag{2.0.19}$$

For future use, we also compute that

$$\nabla_y G(x,y) = -\frac{x-y}{4\pi|x-y|^3} + \frac{1}{4\pi}\frac{R}{|x|}\frac{x^*-y}{|x^*-y|^3}. \tag{2.0.20}$$

Now when $\sigma \in \partial B_R(0)$, (2.0.10) and (2.0.14) imply that

$$|x^* - \sigma| = \frac{R}{|x|}|x - \sigma|. \tag{2.0.21}$$

Therefore, using (2.0.20) and (2.0.21), we compute that

$$\begin{aligned}\nabla_\sigma G(x,\sigma) &= -\frac{x-\sigma}{4\pi|x-\sigma|^3} + \frac{1}{4\pi}\frac{|x|^2}{R^2}\frac{x^*-\sigma}{|x-\sigma|^3} = -\frac{x-\sigma}{4\pi|x-\sigma|^3} + \frac{1}{4\pi}\frac{|x|^2}{R^2}\frac{\frac{R^2}{|x|^2}x-\sigma}{|x-\sigma|^3} \\ &= \frac{\sigma}{4\pi|x-\sigma|^3}\Big(1 - \frac{|x|^2}{R^2}\Big).\end{aligned} \tag{2.0.22}$$

Using (2.0.22) and the fact that $\hat{N}(\sigma) = \frac{1}{R}\sigma$, we deduce

$$\nabla_{\hat{N}(\sigma)} G(x,\sigma) \overset{\text{def}}{=} \nabla_\sigma G(x,\sigma)\cdot\hat{N}(\sigma) = \frac{R^2 - |x|^2}{4\pi R}\frac{1}{|x-\sigma|^3}. \tag{2.0.23}$$

Remark 2.0.2. If the ball were centered at the point $p \in \mathbb{R}^3$ instead of the origin, then the formula (2.0.23) would be replaced with

$$\nabla_{\hat{N}(\sigma)} G(x,\sigma) \overset{\text{def}}{=} \nabla_\sigma G(x,\sigma) \cdot \hat{N}(\sigma) = -\frac{R^2 - |x-p|^2}{4\pi R} \frac{1}{|x-\sigma|^3}. \tag{2.0.24}$$

Let's summarize this by stating a lemma.

Lemma 2.0.1. *The Green function for a ball $B_R(p) \subset \mathbb{R}^3$ is*

$$G(x,y) = -\frac{1}{4\pi|x-y|} + \frac{1}{4\pi} \frac{R}{|x-p| \left| \frac{R^2}{|x-p|^2}(x-p) - (y-p) \right|}, \qquad x \neq p, \tag{2.0.25a}$$

$$G(p,y) = -\frac{1}{4\pi|y-p|} + \frac{1}{4\pi R}. \tag{2.0.25b}$$

Furthermore, if $x \in B_R(p)$ and $\sigma \in \partial B_R(p)$, then

$$\nabla_{\hat{N}(\sigma)} G(x,\sigma) = \frac{R^2 - |x-p|^2}{4\pi R} \frac{1}{|x-\sigma|^3}. \tag{2.0.25c}$$

We can now easily derive a representation formula for solutions to the Laplace equation on a ball.

Theorem 2.1 (Poisson's formula). *Let $B_R(p) \subset \mathbb{R}^3$ be a ball of radius R centered at $p = (p^1, p^2, p^3)$, and let $x = (x^1, x^2, x^3)$ denote a point in $\mathbb{R}^3$. Let $g \in C(\partial B_R(p))$. Then the unique solution $u \in C^2(B_R(p)) \cap C(\overline{B}_R(p))$ of the PDE*

$$\begin{cases} \Delta u(x) = 0, & x \in B_R(p), \\ u(x) = g(x), & x \in \partial B_R(p), \end{cases} \tag{2.0.26}$$

can be represented using the Poisson formula:

$$u(x) = \frac{R^2 - |x-p|^2}{4\pi R} \int_{\partial B_R(p)} \frac{g(\sigma)}{|x-\sigma|^3} \, d\sigma. \tag{2.0.27}$$

Remark 2.0.3. In n dimensions, the formula (2.0.27) gets replaced with

$$u(x) = \frac{R^2 - |x-p|^2}{\omega_n R} \int_{\partial B_R(p)} \frac{g(\sigma)}{|x-\sigma|^n} \, d\sigma, \tag{2.0.28}$$

where as usual, ω_n is the surface area of the unit ball in $\mathbb{R}^n$.

Proof. The identity (2.0.27) follows immediately from Theorem 1.1 and Lemma 2.0.1. □

3. Harnack's inequality

We will now use some of our tools to prove a famous inequality for Harmonic functions. The theorem provides some estimates that place limitations on how slow/fast harmonic functions are allowed to grow.

Theorem 3.1 (**Harnack's inequality**). *Let $B_R(0) \subset \mathbb{R}^n$ be the ball of radius R centered at the origin, and let $u \in C^2(B_R(0)) \cap C(\overline{B}_R(0))$ be the unique solution to (2.0.26). Assume that u is* ***non-negative*** *on $\overline{B}_R(0)$. Then for any $x \in B_R(0)$, we have that*

$$\frac{R^{n-2}(R-|x|)}{(R+|x|)^{n-1}}u(0) \leq u(x) \leq \frac{R^{n-2}(R+|x|)}{(R-|x|)^{n-1}}u(0). \tag{3.0.29}$$

Proof. We'll do the proof for $n = 3$. The basic idea is to combine the Poisson representation formula with simple inequalities and the mean value property. By Theorem 2.1, we have that

$$u(x) = \frac{R^2 - |x|^2}{4\pi R}\int_{\partial B_R(0)} \frac{g(\sigma)}{|x-\sigma|^3}\, d\sigma. \tag{3.0.30}$$

By the triangle inequality, for $\sigma \in \partial B_R(0)$ (i.e. $|\sigma| = R$), we have that $|x| - R \leq |x-\sigma| \leq |x| + R$. Applying the first inequality to (3.0.30), and **using the non-negativity of** g, we deduce that

$$u(x) \leq \frac{R+|x|}{R^2-|x|^2}\frac{1}{4\pi R}\int_{\partial B_R(0)} g(\sigma)\, d\sigma. \tag{3.0.31}$$

Now recall that by the mean value property, we have that

$$u(0) = \frac{1}{4\pi R^2}\int_{\partial B_R(0)} g(\sigma)\, d\sigma. \tag{3.0.32}$$

Thus, combining (3.0.31) and (3.0.32), we have that

$$u(x) \leq \frac{R(R+|x|)}{(R-|x|)^2}u(0), \tag{3.0.33}$$

which implies one of the inequalities in (3.0.29). The other one can be proved similarly using the remaining triangle inequality.

□

We now prove a famous consequence of Harnack's inequality. The statement is also often proved in introductory courses in complex analysis, and it plays a central role in some proofs of the fundamental theorem of algebra.

Corollary 3.0.2 (**Liouville's theorem**). *Suppose that $u \in C^2(\mathbb{R}^n)$ is harmonic on $\mathbb{R}^n$. Assume that there exists a constant M such that $u(x) \geq M$ for all $x \in \mathbb{R}^n$, or such that $u(x) \leq M$ for all $x \in \mathbb{R}^n$. Then u is a constant-valued function.*

Proof. We first consider the case that $u(x) \geq M$. Let $v \overset{\text{def}}{=} u + |M|$. Observe that $v \geq 0$ is harmonic and verifies the hypotheses of Theorem 3.1. Thus, by (3.0.29), if $x \in \mathbb{R}^n$ and R is sufficiently large, we have that

$$\frac{R^{n-2}(R-|x|)}{(R+|x|)^{n-1}}v(0) \leq v(x) \leq \frac{R^{n-2}(R+|x|)}{(R-|x|)^{n-1}}v(0). \tag{3.0.34}$$

Allowing $R \to \infty$ in (3.0.34), we conclude that $v(x) = v(0)$. Thus, v is a constant-valued function (and therefore u is too).

To handle the case $u(x) \leq M$, we simply consider the function $w(x) \stackrel{\text{def}}{=} -u(x) + |M|$ in place of $v(x)$, and we argue as above.

$\square$