

## MATH 18.152 COURSE NOTES - CLASS MEETING # 16

18.152 Introduction to PDEs, Fall 2011

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### Class Meeting # 16: The Fourier Transform on $\mathbb{R}^n$

#### 1. INTRODUCTION TO THE FOURIER TRANSFORM

Earlier in the course, we learned that periodic functions  $f \in L^2([-1, 1])$  (of period 2) can be represented using a Fourier series:

$$(1.0.1) \quad f(x) \overset{\text{“=”}}{=} \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\pi x) + \sum_{m=1}^{\infty} b_m \sin(m\pi x).$$

The “=” sign above is interpreted in the sense of the convergence of the sequence of partial sums associated to the right-hand side in the  $L^2([-1, 1])$  norm. The coefficients  $a_m$  and  $b_m$  represent the “amount of the frequency  $m$ ” that the function  $f$  contains. These coefficients were related to  $f$  itself by

$$(1.0.2a) \quad a_0 = \int_{-1}^1 f(x) dx,$$

$$(1.0.2b) \quad a_m = \int_{-1}^1 f(x) \cos(m\pi x) dx, \quad (m \geq 1),$$

$$(1.0.2c) \quad b_m = \int_{-1}^1 f(x) \sin(m\pi x) dx, \quad (m \geq 1).$$

The *Fourier transform* is a “continuous” version of the formula (1.0.1) for functions defined on the whole space  $\mathbb{R}^n$ . Our goal is to write functions  $f$  defined on  $\mathbb{R}^n$  as a superposition of different frequencies. However, instead of discrete frequencies  $m$ , we will need to use “continuous frequencies”  $\xi$ .

**Definition 1.0.1 (Fourier Transform).** Let  $f \in L^1(\mathbb{R}^n)$ , i.e.,  $\int_{\mathbb{R}^n} |f(x)| d^n x < \infty$ . The Fourier transform of  $f$  is denoted by  $\hat{f}$ , and it is a new function of the frequency variable  $\xi \in \mathbb{R}^n$ . It is defined for each frequency  $\xi$  as follows:

$$(1.0.3) \quad \hat{f}(\xi) \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} d^n x,$$

where  $\cdot$  denotes the Euclidean dot product, i.e., if  $x = (x^1, \dots, x^n)$  and  $\xi = (\xi^1, \dots, \xi^n)$ , then  $\xi \cdot x \stackrel{\text{def}}{=} \sum_{j=1}^n \xi^j x^j$ . In the above formula, recall that if  $r$  is a real number, then  $e^{ir} = \cos r + i \sin r$ .

The formula (1.0.3) is analogous to the formulas (1.0.2a) - (1.0.2c). It provides the “amount of the frequency component”  $\xi$  that  $f$  contains. Later in the course, we will derive an analog of the representation formula (1.0.1).

**Remark 1.0.1.** The Fourier transform can be defined on a much larger class of functions than those that belong to  $L^1$ . However, to make rigorous sense of this fact requires advanced techniques that go beyond this course.

We will also use the following notation.

**Definition 1.0.2 (Inverse Fourier transform).** Given a function  $f(\xi) \in L^1(\mathbb{R}^n)$ , its inverse Fourier transform, which is denoted by  $f^\vee$ , is a new function of  $x$  defined as follows:

$$(1.0.4) \quad f^\vee(x) \stackrel{\text{def}}{=} \hat{f}(-x) \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} f(\xi) e^{2\pi i \xi \cdot x} d^n \xi.$$

The name is motivated as follows: later in the course, we will show that  $(\hat{f})^\vee = f$ . Thus,  $\vee$  is in fact the inverse of the operator  $\wedge$ .

The Fourier transform is very useful in the study of certain PDEs. To use it in the context of PDEs, we will have to understand how the Fourier transform operator interacts with partial derivatives. In order to do this, it is convenient to introduce the following notation, which will simultaneously help us bookkeep when taking repeated derivatives, and when classifying the structure monomials.

**Definition 1.0.3.** If

$$(1.0.5) \quad \vec{\alpha} \stackrel{\text{def}}{=} (\alpha^1, \dots, \alpha^n)$$

is an array of *non-negative* integers, then we define  $\partial_{\vec{\alpha}}$  to be the differential operator

$$(1.0.6) \quad \partial_{\vec{\alpha}} \stackrel{\text{def}}{=} \partial_1^{\alpha^1} \dots \partial_n^{\alpha^n}.$$

Note that  $\partial_{\vec{\alpha}}$  is an operator of order  $|\vec{\alpha}| \stackrel{\text{def}}{=} \alpha^1 + \dots + \alpha^n$ .

If  $x = (x^1, \dots, x^n)$  is an element of  $\mathbb{C}^n$ , then we also define  $x^{\vec{\alpha}}$  to be the monomial

$$(1.0.7) \quad x^{\vec{\alpha}} \stackrel{\text{def}}{=} (x^1)^{\alpha^1} \dots (x^n)^{\alpha^n}.$$

The following function spaces will play an important role in our study of the Fourier transform. Throughout this discussion, the functions  $f$  are allowed to be complex-valued.

**Definition 1.0.4 (Some important function spaces).**

$$(1.0.8) \quad C^k \stackrel{\text{def}}{=} \{f : \mathbb{R}^n \rightarrow \mathbb{C} \mid \partial_{\vec{\alpha}} f \text{ is continuous for } |\vec{\alpha}| \leq k\},$$

$$(1.0.9) \quad C_0 \stackrel{\text{def}}{=} \{f : \mathbb{R}^n \rightarrow \mathbb{C} \mid f \text{ is continuous and } \lim_{|x| \rightarrow \infty} f(x) = 0\}.$$

We also recall the following norm on the space of bounded, continuous functions  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  :

$$(1.0.10) \quad \|f\|_{C_0} \stackrel{\text{def}}{=} \max_{x \in \mathbb{R}^n} |f(x)|.$$

The  $L^2$  norm plays an important role in Fourier analysis. Since  $\hat{f}$  is in general complex-valued we also need to extend the notion of the  $L^2$  inner product to complex-valued functions. This is accomplished in the next definition.

**Definition 1.0.5 (Inner product for complex-valued functions).** Let  $f$  and  $g$  be complex-valued functions defined on  $\mathbb{R}^n$ . We define their complex inner product by

$$(1.0.11) \quad \langle f, g \rangle \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} f(x) \bar{g}(x) d^n x,$$

where  $\bar{g}$  denotes the complex conjugate of  $g$ . That is, if  $g(x) = u(x) + iv(x)$ , where  $u$  and  $v$  are real-valued, then  $\bar{g}(x) \stackrel{\text{def}}{=} u(x) - iv(x)$ .

We also define norm of  $f$  by

$$(1.0.12) \quad \|f\| \stackrel{\text{def}}{=} \langle f, f \rangle^{1/2} \stackrel{\text{def}}{=} \left( \int_{\mathbb{R}^n} |f(x)|^2 d^n x \right)^{1/2}.$$

Note that this is just the standard  $L^2$  norm extended to complex-valued functions.

Note that  $\langle \cdot, \cdot \rangle$  and  $\|\cdot\|$  verify all of the standard properties associated to a complex inner product and its norm:

- $\|f\| \geq 0$  and  $\|f\| = 0$  if and only if  $f = 0$  almost everywhere
- $\langle g, f \rangle = \overline{\langle f, g \rangle}$  (Hermitian symmetry)
- If  $a$  and  $b$  are complex numbers, then  $\langle af + bg, h \rangle = a\langle f, h \rangle + b\langle g, h \rangle$ , and  $\langle f, ag \rangle = \bar{a}\langle f, g \rangle$  (Hermitian linearity)
- $|\langle f, g \rangle| \leq \|f\| \|g\|$  (Cauchy-Schwarz inequality)
- $\|f + g\| \leq \|f\| + \|g\|$  (Triangle Inequality)

## 2. PROPERTIES OF THE FOURIER TRANSFORM

The next lemma illustrates some basic properties of  $\hat{f}$  that hold whenever  $f \in L^1$ .

**Lemma 2.0.1 (Properties of  $\hat{f}$  for  $f \in L^1$ ).** Suppose that  $f \in L^1(\mathbb{R}^n)$ . Then  $\hat{f}$  is a bounded, continuous function and

$$(2.0.13) \quad \|\hat{f}\|_{C_0} \leq \|f\|_{L^1}.$$

*Proof.* Since  $|e^{ir}| = 1$  for all real numbers  $r$ , it follows that for each fixed  $\xi$ , we have

$$(2.0.14) \quad |\hat{f}(\xi)| \leq \int_{\mathbb{R}^n} |f(x) e^{-2\pi i \xi \cdot x}| d^n x \leq \int_{\mathbb{R}^n} |f(x)| d^n x \stackrel{\text{def}}{=} \|f\|_{L^1}.$$

Taking the max over all  $\xi \in \mathbb{R}^n$ , the estimate (2.0.13) thus follows.

We now prove that  $\hat{f}$  is continuous. Given  $\epsilon > 0$ , let  $B_R$  be a ball of radius  $R$  centered at the origin such that the integral of  $|f|$  over its complement  $B_R^c$  is no larger than  $\epsilon$ :

$$(2.0.15) \quad \int_{B_R^c} |f(x)| d^n x \leq \epsilon.$$

It is possible to choose such a ball since  $f \in L^1$ . We then estimate

$$\begin{aligned}
(2.0.16) \quad |\hat{f}(\xi) - \hat{f}(\eta)| &\leq \int_{B_R} |f(x)| |e^{-2\pi i \xi \cdot x} - e^{-2\pi i \eta \cdot x}| d^n x + \int_{B_R^c} |f(x)| \overbrace{|e^{-2\pi i \xi \cdot x} - e^{-2\pi i \eta \cdot x}|}^{\leq 2} d^n x \\
&\leq \int_{B_R} |f(x)| |e^{-2\pi i \xi \cdot x} - e^{-2\pi i \eta \cdot x}| d^n x + 2\epsilon.
\end{aligned}$$

Now since  $e^{-2\pi i r}$  is a uniformly continuous function of the real number  $r$  on any compact set, if  $|\xi - \eta|$  is sufficiently small, then we can ensure that  $\max_{x \in B_R} |e^{-2\pi i \xi \cdot x} - e^{-2\pi i \eta \cdot x}| \leq \epsilon$ . We then conclude that the final integral over  $B_R$  on the right-hand side of (2.0.16) will be no larger than

$$(2.0.17) \quad \max_{x \in B_R} |e^{-2\pi i \xi \cdot x} - e^{-2\pi i \eta \cdot x}| \int_{B_R} |f(x)| d^n x \leq \epsilon \int_{\mathbb{R}^n} |f(x)| d^n x \stackrel{\text{def}}{=} \epsilon \|f\|_{L^1}.$$

Thus, in total, we have shown that if  $|\xi - \eta|$  is sufficiently small, then  $|\hat{f}(\xi) - \hat{f}(\eta)| \leq \epsilon \|f\|_{L^1} + 2\epsilon$ . Since such an estimate holds for all  $\epsilon > 0$ ,  $\hat{f}$  is continuous by definition.  $\square$

It is helpful to introduce notation to indicate that a function has been translated.

**Definition 2.0.6 (Translation of a function).** If  $\mathbb{R}^n \rightarrow \mathbb{C}$  is a function and  $y \in \mathbb{R}^n$  is any point, then we define the translated function  $\tau_y f$  by

$$(2.0.18) \quad \tau_y f(x) \stackrel{\text{def}}{=} f(x - y).$$

The next theorem collects together some very important properties of the Fourier transform. In particular, it illustrates how the Fourier transform interacts with translations, derivatives, multiplication by polynomials, products, convolutions, and complex conjugates.

**Theorem 2.1 (Important properties of the Fourier transform).** Assume that  $f, g \in L^1(\mathbb{R}^n)$ , and let  $t \in \mathbb{R}$ . Then

$$(2.0.19a) \quad (\tau_y f)^\wedge(\xi) = e^{-2\pi i \xi \cdot y} \hat{f}(\xi),$$

$$(2.0.19b) \quad \hat{h}(\xi) = \tau_\eta \hat{f}(\xi) \quad \text{if } h(x) \stackrel{\text{def}}{=} e^{2\pi i \eta \cdot x} f(x),$$

$$(2.0.19c) \quad \hat{h}(\xi) = t^n \hat{f}(t\xi) \quad \text{if } h(x) \stackrel{\text{def}}{=} f(t^{-1}x),$$

$$(2.0.19d) \quad (f * g)^\wedge(\xi) = \hat{f}(\xi) \hat{g}(\xi),$$

$$(2.0.19e) \quad \text{If } x^{\vec{\alpha}} f \in L^1 \text{ for } |\vec{\alpha}| \leq k, \text{ then } \hat{f} \in C^k \text{ and } \partial_{\vec{\alpha}} \hat{f}(\xi) = [(-2\pi i x)^{\vec{\alpha}} f(x)]^\wedge(\xi),$$

$$(2.0.19f) \quad \text{If } f \in C^k, \partial_{\vec{\alpha}} f \in L^1 \text{ for } |\vec{\alpha}| \leq k, \text{ and } \partial_{\vec{\alpha}} f \in C_0 \text{ for } |\vec{\alpha}| \leq k - 1, \text{ then } (\partial_{\vec{\alpha}} f)^\wedge(\xi) = (2\pi i \xi)^{\vec{\alpha}} \hat{f}(\xi),$$

$$(2.0.19g) \quad \hat{\bar{f}}(\xi) = (\bar{f})^\vee(\xi) \text{ and } \overline{(\hat{f})^\vee}(\xi) = (\bar{f})^\wedge(\xi).$$

Above,  $\bar{f}$  denotes the complex conjugate of  $f$ ; i.e., if  $f = u + iv$ , where  $u$  and  $v$  are real-valued, then  $\bar{f} = u - iv$ .

*Proof.* To prove (2.0.19a), we make the change of variables  $z = x - y$ ,  $d^n z = d^n x$  and calculate that

$$(2.0.20) \quad (\tau_y f)^\wedge(\xi) \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} f(x - y) e^{-2\pi i x \cdot \xi} d^n x = \int_{\mathbb{R}^n} f(z) e^{-2\pi i(z+y) \cdot \xi} d^n z = e^{-2\pi i y \cdot \xi} \int_{\mathbb{R}^n} f(z) e^{-2\pi i z \cdot \xi} d^n z \stackrel{\text{def}}{=} e^{-2\pi i y \cdot \xi} \hat{f}(\xi).$$

To prove (2.0.19b), we calculate that

$$(2.0.21) \quad \hat{h}(\xi) \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} e^{2\pi i \eta \cdot x} f(x) e^{-2\pi i x \cdot \xi} d^n x = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot (\xi - \eta)} d^n x \stackrel{\text{def}}{=} \hat{f}(\xi - \eta) \stackrel{\text{def}}{=} \tau_\eta \hat{f}(\xi).$$

To prove (2.0.19c), we make the change of variables  $y = t^{-1}x$ ,  $d^n y = t^{-n}d^n x$  to deduce that

$$(2.0.22) \quad \begin{aligned} \hat{h}(\xi) &\stackrel{\text{def}}{=} \int_{\mathbb{R}^n} f(t^{-1}x) e^{-2\pi i x \cdot \xi} d^n x \\ &= \int_{\mathbb{R}^n} f(y) e^{-2\pi i y \cdot t\xi} t^n d^n y \\ &\stackrel{\text{def}}{=} t^n \hat{f}(t\xi). \end{aligned}$$

To prove (2.0.19d), we use the definition of convolution, (2.0.19a), and Fubini's theorem to deduce that

$$(2.0.23) \quad \begin{aligned} (f * g)^\wedge(\xi) &\stackrel{\text{def}}{=} \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} \left( \int_{\mathbb{R}^n} f(x - y) g(y) d^n y \right) d^n x = \int_{\mathbb{R}^n} g(y) \underbrace{\left( \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x - y) d^n x \right)}_{e^{-2\pi i \xi \cdot y} \hat{f}(\xi)} d^n y \\ &= \hat{f}(\xi) \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot y} g(y) d^n y \stackrel{\text{def}}{=} \hat{f}(\xi) \hat{g}(\xi). \end{aligned}$$

To prove (2.0.19e), we differentiate under the integral in the definition of  $\hat{f}(\xi)$  to deduce that

$$(2.0.24) \quad \partial_{\bar{\alpha}}^{(\xi)} \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) \partial_{\bar{\alpha}}^{(\xi)} e^{-2\pi i x \cdot \xi} d^n x = \int_{\mathbb{R}^n} f(x) (-2\pi i x)^{\bar{\alpha}} e^{-2\pi i x \cdot \xi} d^n x \stackrel{\text{def}}{=} [(-2\pi i x)^{\bar{\alpha}} f(x)]^\wedge(\xi).$$

To prove (2.0.19f), we integrate by parts  $|\bar{\alpha}|$  times and use the hypotheses on  $f$  to discard the boundary terms at infinity, thus concluding that

$$(2.0.25) \quad \begin{aligned} (\partial_{\bar{\alpha}} f)^\wedge(\xi) &\stackrel{\text{def}}{=} \int_{\mathbb{R}^n} \partial_{\bar{\alpha}} f(x) e^{-2\pi i x \cdot \xi} d^n x = \int_{\mathbb{R}^n} f(x) (-1)^{|\bar{\alpha}|} \partial_{\bar{\alpha}}^{(x)} e^{-2\pi i x \cdot \xi} d^n x \\ &= \int_{\mathbb{R}^n} f(x) (2\pi i \xi)^{\bar{\alpha}} e^{-2\pi i x \cdot \xi} d^n x \stackrel{\text{def}}{=} (2\pi i \xi)^{\bar{\alpha}} \hat{f}(\xi). \end{aligned}$$

To deduce the first relation in (2.0.19g), we compute that

(2.0.26)

$$\widehat{f}(\xi) \stackrel{\text{def}}{=} \overline{\int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} d^n x} = \int_{\mathbb{R}^n} \overline{f(x)} e^{-2\pi i x \cdot \xi} d^n x = \int_{\mathbb{R}^n} \overline{f(x)} e^{2\pi i x \cdot \xi} d^n x \stackrel{\text{def}}{=} \widehat{\bar{f}}(-\xi) \stackrel{\text{def}}{=} (\bar{f})^\vee(\xi).$$

The second relation in (2.0.19g) can be shown using similar reasoning. □

(2.0.19e) roughly shows that if  $f$  decays very rapidly at infinity, then  $\widehat{f}$  is very differentiable. Similarly, (2.0.19f) roughly shows that if  $f$  is very differentiable with rapidly decaying derivatives, then  $\widehat{f}$  also rapidly decays. The Fourier transform thus connects the decay properties of  $f$  to the differentiability properties of  $\widehat{f}$ , and vice versa. In the next proposition, we provide a specific example of these phenomena. More precisely, the next proposition shows that the Fourier transform of a smooth, compactly supported function is itself smooth and rapidly decaying at infinity.

**Proposition 2.0.2.** *Let  $f \in C_c^\infty(\mathbb{R}^n)$ , i.e.,  $f$  is a smooth, compactly supported function. Then  $\widehat{f}$  is smooth and “rapidly decaying at infinity” in the following sense: for each  $N \geq 0$ , there exists a constant  $C_N > 0$  such that*

$$(2.0.27) \quad |\widehat{f}(\xi)| \leq C_N (1 + |\xi|)^{-N}.$$

Furthermore, an estimate similar to (2.0.27) holds (with possibly different constants) for all of the derivatives  $|\partial_{\vec{\beta}} \widehat{f}(\xi)|$ .

In particular,  $\widehat{f} \in L^1$  :

$$(2.0.28) \quad \|\widehat{f}(\xi)\|_{L^1} \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} |\widehat{f}(\xi)| d^n \xi < \infty,$$

and similarly for  $\partial_{\vec{\beta}} \widehat{f}$ , where  $\vec{\beta}$  is any derivative multi-index.

*Proof.* Using (2.0.19e) and the fact that  $f$  is compactly supported (and hence  $x^{\vec{\alpha}} f \in L^1$ ), we see that  $\widehat{f}$  is smooth.

To prove (2.0.27), we use (2.0.19f), (2.0.13), and the fact that  $\|\partial_{\vec{\alpha}} f\|_{L^1} < \infty$  for any differential operator  $\partial_{\vec{\alpha}}$  to deduce that

$$(2.0.29) \quad |(2\pi i \xi)^{\vec{\alpha}} \widehat{f}(\xi)| = |(\partial_{\vec{\alpha}} f)^\wedge(\xi)| \leq \|\partial_{\vec{\alpha}} f\|_{L^1} = C_{\vec{\alpha}},$$

where  $C_{\vec{\alpha}}$  is a constant depending on  $\vec{\alpha}$ . In particular, if  $M \geq 0$  is an integer, then by applying (2.0.29) to the differential operator  $\Delta^M \stackrel{\text{def}}{=} (\sum_{i=1}^n \partial_i^2)^M$  (i.e.,  $|(2\pi i)^{2M} \underbrace{(\sum_{i=1}^n (\xi^i)^2)^M}_{|\xi|^{2M}} \widehat{f}(\xi)| =$

$|(\Delta^M f)^\wedge(\xi)| \leq C_M$ ), it follows that

$$(2.0.30) \quad (2\pi |\xi|)^{2M} |\widehat{f}(\xi)| \leq C_M$$

for some constant  $C_M > 0$ . It is easy to see that an estimate of the form (2.0.27) follows from (2.0.30).

(2.0.28) follows from (2.0.27) and the fact that

$$(2.0.31) \quad \int_{\mathbb{R}^n} \frac{1}{(1 + |\xi|)^{n+1}} d^n \xi < \infty.$$

To see that (2.0.31) holds, perform the integration using spherical coordinates on  $\mathbb{R}^n$  :

$$(2.0.32) \quad \int_{\mathbb{R}^n} \frac{1}{(1 + |\xi|)^{n+1}} d^n \xi = \omega_n \int_{\rho=0}^{\infty} \frac{\rho^{n-1}}{(1 + \rho)^{n+1}} d\rho,$$

where  $\rho \stackrel{\text{def}}{=} |\xi| \stackrel{\text{def}}{=} \sqrt{\sum_{j=1}^n (\xi^j)^2}$  is the radial variable on  $\mathbb{R}^n$ , and  $\omega_n$  is the surface area of the unit ball in  $\mathbb{R}^n$ . By a simple comparison estimate, it is easy to see that the integral on the right-hand side of (2.0.32) converges (the integrand behaves like 0 near  $\rho = 0$ , and like  $\frac{1}{\rho^2}$  near  $\infty$ ).

To show that similar results hold for  $\partial_{\vec{\beta}} \hat{f}$ , we first use (2.0.19e) to conclude that

$$(2.0.33) \quad \partial_{\vec{\beta}} \hat{f}(\xi) = [(-2\pi i x)^{\vec{\beta}} f(x)]^{\wedge}(\xi).$$

Furthermore, the function  $(-2\pi i x)^{\vec{\beta}} f(x)$  also satisfies the hypotheses of the proposition. We can therefore repeat the above arguments with  $\partial_{\vec{\beta}} \hat{f}$  in place of  $\hat{f}$  and  $(-2\pi i x)^{\vec{\beta}} f(x)$  in place of  $f$ . □

### 3. GAUSSIANS

One of the most important classes of functions in Fourier theory is the class of Gaussians. The next proposition shows that this class interacts very nicely with the Fourier transform.

**Proposition 3.0.3 (The Fourier transform of a Gaussian is another Gaussian).** *Let  $f(x) = \exp(-\pi z|x|^2)$ , where  $z = a + ib$  is a complex number,  $a, b \in \mathbb{R}$ ,  $a > 0$ ,  $x = (x^1, \dots, x^n) \in \mathbb{R}^n$ , and  $|x|^2 = \sum_{j=1}^n (x^j)^2$ . Then*

$$(3.0.34) \quad \hat{f}(\xi) = z^{-n/2} \exp(-\pi|\xi|^2/z).$$

*Proof.* We consider only the case  $b = 0$ , so that  $z = a$ . The cases  $b \neq 0$  would follow from an argument similar to the one we give below but requiring a few additional technical details. We first address the case  $n = 1$ . Then by properties (2.0.19e)-(2.0.19f) of Theorem 2.1, we have that

$$(3.0.35) \quad \hat{f}'(\xi) = (-2\pi i x e^{-a\pi x^2})^{\wedge}(\xi) = \frac{i}{a} \left( \frac{d}{dx} e^{-a\pi x^2} \right)^{\wedge}(\xi) = \frac{i}{a} 2\pi i \xi \hat{f}(\xi) = \frac{-2\pi}{a} \xi \hat{f}(\xi).$$

We can view (3.0.35) as

$$(3.0.36) \quad \frac{d}{d\xi} \ln \hat{f} = \frac{-2\pi}{a} \xi.$$

Integrating (3.0.36) with respect to  $\xi$  and then exponentiating both sides, we conclude that

$$(3.0.37) \quad \hat{f}(\xi) = C \exp(-\pi \xi^2/a.)$$

Furthermore, the constant  $C$  clearly must be equal to  $\hat{f}(0)$ .

We now compute  $\hat{f}(0)$  :

$$(3.0.38) \quad \hat{f}(0) \stackrel{\text{def}}{=} \int_{\mathbb{R}} e^{-\pi a x^2} \overbrace{e^{-2\pi i \xi 0}}^1 dx = a^{-1/2}.$$

Note that you have previously calculated this integral in your homework. Combining (3.0.36) and (3.0.38), we arrive at the desired expression (3.0.34) in the case  $n = 1$ .

To treat the case of general  $n$ , we note that the properties of the exponential function and the Fubini theorem together allow us to reduce it to the case of  $n = 1$  :

$$(3.0.39) \quad \begin{aligned} \hat{f}(\xi) &= \int_{\mathbb{R}^n} \exp(-\pi a|x|^2)\exp(-2\pi i\xi \cdot x) d^n x \\ &= \int_{\mathbb{R}^n} \exp\left(-\pi a \sum_{k=1}^n (x^k)^2\right)\exp\left(-2\pi i \sum_{j=1}^n \xi^j x^j\right) d^n x \\ &= \int_{\mathbb{R}^n} \prod_{j=1}^n \left\{ \exp(-\pi a(x^j)^2)\exp(-2\pi i\xi^j x^j) \right\} d^n x \\ &= \prod_{j=1}^n \left\{ \int_{\mathbb{R}} \exp(-\pi a(x^j)^2)\exp(-2\pi i\xi^j x^j) dx^j \right\} \\ &= \prod_{j=1}^n a^{-1/2}\exp(-\pi(\xi^j)^2/a) \\ &= a^{-n/2}\exp\left(-\pi a^{-1} \sum_{j=1}^n (\xi^j)^2\right) \\ &= a^{-n/2}\exp(-\pi|\xi|^2/a). \end{aligned}$$

We have thus shown (3.0.34). □

#### 4. FOURIER INVERSION AND THE PLANCHEREL THEOREM

The next lemma is very important. It shows that the Fourier transform interacts nicely with the  $L^2$  inner product.

**Lemma 4.0.4 (Interaction of the Fourier transform with the  $L^2$  inner product).** *Assume that  $f, g \in L^1$ . Then*

$$(4.0.40) \quad \int_{\mathbb{R}^n} \hat{f}(x)g(x) d^n x = \int_{\mathbb{R}^n} f(x)\hat{g}(x) d^n x.$$

*Alternatively, in terms of the complex  $L^2$  inner product, we have that*

$$(4.0.41) \quad \langle \hat{f}, g \rangle = \langle f, g^\vee \rangle.$$

*Proof.* Using the definition of the Fourier transform and Fubini's theorem, the left-hand side of (4.0.40) is equal to

$$(4.0.42) \quad \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\xi)g(x)e^{-2\pi i\xi \cdot x} d^n\xi d^n x.$$

By the same reasoning, this is also equal to the right-hand side of (4.0.40).

To obtain (4.0.41), simply replace  $g$  with  $\bar{g}$  in the identity (4.0.40) and use property (2.0.19g).  $\square$

The next theorem is central to Fourier analysis. It shows that the operators  $\wedge$  and  $\vee$  are inverses of each other whenever  $f$  and  $\hat{f}$  are nice functions.

**Theorem 4.1 (Fourier inversion theorem).** *Suppose that  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  is a continuous function, that  $f \in L^1$ , and that  $\hat{f} \in L^1$ . Then*

$$(4.0.43) \quad (\hat{f})^\vee = (f^\vee)^\wedge = f.$$

*That is, the operators  $\wedge$  and  $\vee$  are inverses of each other.*

*Proof.* We first note that

$$(4.0.44) \quad (\hat{f})^\vee(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} f(y)e^{-2\pi iy \cdot \xi} d^n y \right\} e^{2\pi ix \cdot \xi} d^n \xi.$$

Note that the integral in (4.0.44) is not absolutely convergent when viewed as a function of  $(y, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ . Thus, our proof of (4.0.43) will involve a slightly delicate limiting procedure that makes use of the auxiliary function

$$(4.0.45) \quad \phi(t, \xi) \stackrel{\text{def}}{=} \exp(-\pi t^2|\xi|^2 + 2\pi i\xi \cdot x).$$

Note that (2.0.19b) and Proposition 3.0.3 together imply that

$$(4.0.46) \quad \hat{\phi}(y) = t^{-n} \exp(-\pi|x - y|^2/t^2) \stackrel{\text{def}}{=} \Gamma(t, x - y),$$

where

$$(4.0.47) \quad \Gamma(t; y) \stackrel{\text{def}}{=} t^{-n} \exp(-\pi|y|^2/t^2).$$

Also note that  $\Gamma(t, y)$  is just the fundamental solution of the heat equation with diffusion constant  $D = \frac{1}{4\pi}$ . In particular, we previously showed in our study of the heat equation that

$$(4.0.48) \quad \int_{\mathbb{R}^n} \Gamma(t, y) d^n y = 1$$

for all  $t > 0$ . We now compute that

$$\begin{aligned}
(4.0.49) \quad (\Gamma(t, \cdot) * f)(x) &\stackrel{\text{def}}{=} \int_{\mathbb{R}^n} \Gamma(t, x - y) f(y) d^n y \\
&= \int_{\mathbb{R}^n} \hat{\phi}(t, y) f(y) d^n y \\
&= \int_{\mathbb{R}^n} \phi(t, \xi) \hat{f}(\xi) d^n \xi \\
&= \int_{\mathbb{R}^n} \exp(-\pi t^2 |\xi|^2) \hat{f}(\xi) \exp(2\pi i \xi \cdot x) d^n \xi
\end{aligned}$$

During our study of the heat equation, we showed that the left-hand side of (4.0.49) converges to  $f(x)$  as  $t \downarrow 0$ . To complete the proof of the theorem, it remains to show that the right-hand side converges to

$$(4.0.50) \quad \int_{\mathbb{R}^n} \hat{f}(\xi) \exp(2\pi i \xi \cdot x) d^n \xi \stackrel{\text{def}}{=} (\hat{f})^\vee(x) \stackrel{\text{def}}{=} (\hat{f})^\wedge(-x)$$

as  $t \downarrow 0$ . To this end, given any number  $\epsilon > 0$ , choose a ball  $B_R$  of radius  $R$  centered at the origin such that

$$(4.0.51) \quad \int_{B_R^c} |\hat{f}(\xi)| d^n \xi \leq \epsilon.$$

Above,  $B_R^c$  denotes the complement of the ball. It is possible to choose such a ball since  $\hat{f} \in L^1$ . We then estimate

$$\begin{aligned}
(4.0.52) \quad &\left| \int_{\mathbb{R}^n} \exp(-\pi t^2 |\xi|^2) \hat{f}(\xi) \exp(2\pi i \xi \cdot x) d^n \xi - \hat{f}^\vee(x) \right| \\
&\stackrel{\text{def}}{=} \left| \int_{\mathbb{R}^n} \exp(-\pi t^2 |\xi|^2) \hat{f}(\xi) \exp(2\pi i \xi \cdot x) d^n \xi - \int_{\mathbb{R}^n} \hat{f}(\xi) \exp(2\pi i \xi \cdot x) d^n \xi \right| \\
&\leq \int_{\mathbb{R}^n} |\exp(-\pi t^2 |\xi|^2) - 1| |\hat{f}(\xi)| d^n \xi \\
&\leq \max_{\xi \in B_R} |\exp(-\pi t^2 |\xi|^2) - 1| \int_{B_R} |\hat{f}(\xi)| d^n \xi + \int_{B_R^c} \overbrace{|\exp(-\pi t^2 |\xi|^2) - 1|}^{\leq 1} |\hat{f}(\xi)| d^n \xi \\
&\leq \max_{\xi \in B_R} |\exp(-\pi t^2 |\xi|^2) - 1| \|\hat{f}\|_{L^1} + \int_{B_R^c} |\hat{f}(\xi)| d^n \xi \\
&\leq \max_{\xi \in B_R} |\exp(-\pi t^2 |\xi|^2) - 1| \|\hat{f}\|_{L^1} + \epsilon.
\end{aligned}$$

As  $t \downarrow 0$ , the first term on the right-hand side of (4.0.52) converges to 0. In particular, if  $t$  is sufficiently small, then the right-hand side of (4.0.52) will be no larger than  $2\epsilon$ . Since this holds for any  $\epsilon > 0$ , we have thus shown that the right-hand side of (4.0.49) converges to the expression (4.0.50) as  $t \downarrow 0$ , i.e., that it converges to  $(\hat{f})^\vee(x)$ . Since, as we have previously noted, the left-hand side of (4.0.49) converges to  $f(x)$  as  $t \downarrow 0$ , we have thus shown that  $(\hat{f})^\vee(x) = f(x)$ .

It can similarly be shown that  $(f^\vee)^\wedge(x) = f(x)$ . This completes the proof of (4.0.43).

□

The next theorem plays a central role in many areas of PDE and analysis. It shows that the Fourier transform preserves the  $L^2$  norm of functions.

**Theorem 4.2 (The Plancherel theorem).** *Suppose that  $f, g : \mathbb{R}^n \rightarrow \mathbb{C}$  are continuous functions, that  $f, g \in L^1 \cap L^2$ , and that  $\hat{f}, \hat{g} \in L^1$ . Then  $\hat{f}, \hat{g} \in L^2$ , and*

$$(4.0.53) \quad \langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle,$$

*i.e., the Fourier transform preserves the  $L^2$  inner product. In particular, by setting  $f = g$ , it follows from (4.0.53) that*

$$(4.0.54) \quad \|f\|_{L^2} = \|\hat{f}\|_{L^2}.$$

*Proof.* By applying (4.0.41) with  $g$  replaced by  $\hat{g}$ , we have that

$$(4.0.55) \quad \langle \hat{f}, \hat{g} \rangle = \langle f, (\hat{g})^\vee \rangle.$$

By the Fourier inversion theorem (i.e. Theorem 4.1), we have that  $(\hat{g})^\vee = g$ , and so the right-hand side of (4.0.55) is equal to

$$(4.0.56) \quad \langle f, g \rangle.$$

We have thus shown (4.0.53).

□

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