MATH 18.152 COURSE NOTES - CLASS MEETING # 24

18.152 Introduction to PDEs, Fall 2011 Professor: Jared Speck

Class Meeting # 24: Transport Equations and Burger's Equation

In these notes, we introduce a class of evolution PDEs known as transport equations. Such equations arise in a physical context whenever a quantity is "transported" in a certain direction. Some important physical examples include the mass density flow for an incompressible fluid, and the Boltzmann equation of kinetic theory. We discuss both linear transport equations and a famous nonlinear transport equation known as Burger's equation. One of our major goals is to show that in contrast to the case of linear PDEs, solutions to Burger's equations can develop singularities in finite time.

1. Transport Equations

Linear homogeneous transport equations are PDEs of the form

$$X^\mu \partial_\mu u = 0, \tag{1.0.1}$$

where $(x^0, x^1, \cdots, x^n)$ are coordinates on $\mathbb{R}^{1+n}$ and $X(x^0, \cdots, x^n)$ is a vectorfield on $\mathbb{R}^{1+n}$. As we will soon see, the transport equation is closely connected to the following system of ODEs for the unknowns γ^μ:

$$\frac{d}{ds}\gamma^\mu(s) = X^\mu(\gamma^0(s), \gamma^1(s), \cdots, \gamma^n(s)), \qquad (\mu = 0, 1, \cdots, n). \tag{1.0.2}$$

Given initial conditions $\gamma^\mu(0)$, the solutions to (1.0.2) are curves $\gamma : I \to \mathbb{R}^{1+n}$, where I is an interval. These curves are known as the *integral curves* of the vectorfield X. They are also known as the *characteristic curves* associated to the PDE (1.0.1). The next proposition clarifies the connection between the transport equation (1.0.1) and its characteristic curves.

Proposition 1.0.1 (Connection between transport equations and ODEs). *If u solves the transport equation* (1.0.1)*, then u is constant along the integral curves of X. More precisely, if $\gamma(s)$ is any solution to* (1.0.2)*, then*

$$\frac{d}{ds} u\big(\gamma^0(s), \cdots, \gamma^n(s)\big) = 0. \tag{1.0.3}$$

Proof. Using the chain rule, (1.0.2), and (1.0.1), we have that

$$\begin{aligned} \frac{d}{ds} u\big(\gamma^0(s), \cdots, \gamma^n(s)\big) &= \sum_{\mu=0}^{n} \big(\frac{\partial}{\partial x^\mu} u\big)|_{\gamma(s)} \frac{d}{ds}\gamma^\mu(s) \\ &= \sum_{\mu=0}^{n} \big(\frac{\partial}{\partial x^\mu} u\big)|_{\gamma(s)} X^\mu(\gamma(s)) = (X^\mu \partial_\mu u)|_{\gamma(s)} = 0. \end{aligned} \tag{1.0.4}$$

$\square$

1.1. **Constant vectorfields.** Let's consider a very special case of (1.0.1) in which the components of X are constant. That is, we assume that

$$X = (\overline{X}^0, \overline{X}^1, \cdots, \overline{X}^n) \tag{1.1.1}$$

where the $\overline{X}^\mu$ are constants independent of $(x^0, \cdots, x^n)$.

In this case, the solutions to the system (1.0.2) of ODEs are the *straight lines*

$$\gamma(s) = \mathring{\gamma} + sX, \tag{1.1.2}$$

where $\mathring{\gamma} = \gamma(0)$ is a constant vector.

For concreteness, let's also assume that

$$\overline{X}^0 = 1 \tag{1.1.3}$$

and as usual, let's use the alternate notation $x^0 = t$. Let's assume that we are given *Cauchy data* for u on the hypersurface $\{t = 0\} \times \mathbb{R}^n$:

$$u(0, x^1, \ldots, x^n) = f(x^1, \cdots, x^n), \tag{1.1.4}$$

where f is a function on $\mathbb{R}^n$. We now note that

$$(t, x^1, \cdots, x^n) = (0, x^1 - t\overline{X}^1, \cdots, x^n - t\overline{X}^n) + tX, \tag{1.1.5}$$

which implies that the spacetime point $(t, x^1, \cdots, x^n)$ lies on the characteristic curve $\gamma(t)$ passing through the "initial" point $(0, x^1 - t\overline{X}^1, \cdots, x^n - t\overline{X}^n) \subset \{t = 0\} \times \mathbb{R}^n$. Therefore, by Proposition 1.0.1, we have that

$$u(0, x^1, \ldots, x^n) = f(x^1 - t\overline{X}^1, \cdots, x^n - t\overline{X}^n), \tag{1.1.6}$$

and we have *explicitly solved* the PDE (1.0.1).

2. A Nonlinear Scalar PDE: Burger's (Inviscid) Equation

Burger's equation is a simple nonlinear PDE in 1+1 dimensions. It is often used to illustrate some important features of (some) nonlinear PDEs. As we will see, it can be viewed as a nonlinear version of the transport equation. Our main goal in these next two sections is to illustrate a phenomenon not found in linear PDEs : the formation of a singularity in the solution.

Burger's equation is the following PDE for the function $u(t, x)$:

$$\partial_t u + u\partial_x u = 0, \qquad (t, x) \in [0, \infty) \times \mathbb{R}. \tag{2.0.7}$$

As we will see, the Cauchy problem (i.e., the initial value problem in which the datum $u(0, x)$ is prescribed) for (2.0.7) is well-posed.

Equation (2.0.7) is a simple example of a nonlinear *conservation law.* More precisely, the next proposition shows that under suitable assumptions, the spatial L^2 norm of solutions to (2.0.7) is preserved in time.

Proposition 2.0.1 (Burger's equation is a conservation law). *Let $T \geq 0$, and let $u(t,x)$ be a C^1 solution to* (2.0.7) *on $S_T \overset{def}{=} [0,T] \times \mathbb{R}$. Assume that for each fixed $t \in [0,T]$, we have that $\lim_{x \to \pm\infty} u(t,x) = 0$. Then for $(t,x) \in S_T$, we have that*

$$\int_{\mathbb{R}} u^2(t,x)\,dx = \int_{\mathbb{R}} u^2(0,x)\,dx, \tag{2.0.8}$$

i.e., the spatial L^2 norm of $u(t,\cdot)$ is preserved in time.

Proof. Multiplying both sides of (2.0.7) by u, we deduce that

$$\frac{1}{2}\partial_t(u^2) + \frac{1}{3}\partial_x(u^3) = 0. \tag{2.0.9}$$

Integrating (2.0.9) over $\mathbb{R}$, using the Fundamental Theorem of calculus and the assumption on the behavior of $u(t,x)$ as $x \to \pm\infty$, and "un-differentiating" under the integral, we deduce that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} |u(t,x)|^2\,dx = 0. \tag{2.0.10}$$

The proposition now follows from (2.0.10).

□

Notice that (2.0.7) can be viewed as as a transport equation whose speed and direction depend on the solution u itself. As in the case of transport equations, we can define the characteristic curves associated to a solution of (2.0.7).

Definition 2.0.1. Let u be a solution of (2.0.7). The characteristic curves associated to u are the solutions to the following system of ODEs:

$$\frac{d}{ds}\gamma^0 = 1, \tag{2.0.11a}$$

$$\frac{d}{ds}\gamma^1 = u \circ \gamma = u(\gamma^0(s), \gamma^1(s)). \tag{2.0.11b}$$

Remark 2.0.1. Equation (2.0.11a) shows that $\gamma^0(s) = s + c$, where c is a constant. There is no loss of generality in parameterizing the curve with the constant c set equal to 0.

The next two propositions are essential for our analysis of Burger's equation.

Proposition 2.0.2 (Burger solutions are constant along characteristics). *C^1 solutions to* (2.0.7) *are constant along the characteristic curves* (2.0.11a) - (2.0.11b).

Proof. Using the chain rule and the equations (2.0.7), (2.0.11a) - (2.0.11b), we compute that

$$\frac{d}{ds}[u \circ \gamma(s)] = (\partial_t u)|_\gamma \frac{d}{ds}\gamma^0 + (\partial_x u)|_\gamma \frac{d}{ds}\gamma^1 = (\partial_t u)|_\gamma + u|_\gamma (\partial_x u)|_\gamma = 0. \tag{2.0.12}$$

□

Proposition 2.0.3 (Burger characteristics are straight lines). *The characteristic curves* (2.0.11a) - (2.0.11b) *are straight lines in $\mathbb{R}^{1+1}$.*

Proof. It clearly follows from (2.0.11a) that

$$\frac{d^2}{ds^2}\gamma^0(s) = 0. \tag{2.0.13}$$

Furthermore, using the ODE (2.0.11b) and the computation (2.0.12), we compute that

$$\frac{d^2}{ds^2}\gamma^1(s) = \frac{d}{ds}[u \circ \gamma(s)] = 0. \tag{2.0.14}$$

We have thus shown that $\frac{d^2}{ds^2}\gamma^\mu(s) = 0$ for $\mu = 0, 1$. Thus, the curve γ has **0** acceleration, and is therefore a straight line.

□

3. "Solving" Burger's equation

Using the propositions from the previous section, will now exhibit an implicit solution to the following initial value problem for Burger's equation:

$$\begin{aligned} \partial_t u + u\partial_x u &= 0, \qquad (t, x) \in [0, \infty) \times \mathbb{R}, \\ u(0, x) &= f(x), \qquad x \in \mathbb{R}. \end{aligned} \tag{3.0.15}$$

Theorem 3.1. *Let u be a C^1 solution to (3.0.15), and let (t, x) be a spacetime point. With (t, x) fixed, assume that the implicit equation $x = p + f(p)t$ in the unknown p has a* unique *solution. Then*

$$u(t, x) = f(p). \tag{3.0.16}$$

Proof. Let $\gamma(s) = (\gamma^0(s), \gamma^1(s))$ denote the characteristic curve passing through the Cartesian (t, x) spacetime point $(0, p)$ when $s = 0$, i.e., $(\gamma^0(0), \gamma^1(0)) = (0, p)$. According to the ODEs (2.0.11a) - (2.0.11b) and Proposition 2.0.3, $\gamma(s)$ is a straight line with constant "t/x" slope $\frac{\dot{\gamma}^0(0)}{\dot{\gamma}^1(0)} = \frac{1}{f(p)}$. It therefore follows that

$$\gamma^0(s) = s, \tag{3.0.17}$$

$$\gamma^1(s) = p + f(p)s. \tag{3.0.18}$$

Consequently, by Proposition 2.0.2, we have that

$$u(s, p + f(p)s) = u(0, p) = f(p). \tag{3.0.19}$$

Equation (3.0.16) thus follows. □

4. Formation of Singularities

Proposition 2.0.1 shows that the spatial L^2 norm of nice solutions to Burger's equation is preserved in time. This conserved quantity suggests that the solution can never grow large and therefore that the solution should exist for all time. However, this intuition is false! The next theorem shows that even though the L^2 norm is preserved, the solution can develop a singularity in finite time, *even if the initial datum f is very small and very nice.*

Theorem 4.1 (Sharp Characterization of Singularity Formation in Burger's Equation). *Let $f \in C^1(\mathbb{R})$ be initial data for Burger's equation* (3.0.15)*. Then the corresponding solution $u(t,x)$ remains C^1 for all $(t,x) \in [0,\infty) \times \mathbb{R}$ if and only if $f'(x) \geq 0$ holds for all $x \in \mathbb{R}$.*

Proof. Suppose that there exists a point x_0 such that $f'(x_0) < 0$. Then there exists a nearby point $x_1 > x_0$ with $f(x_1) < f(x_0)$. Let $\gamma_{(x_i)}(s)$ denote the characteristic curve passing through the spacetime point $(0, x_i)$ at $s = 0$. Then by Proposition 2.0.2, $u \circ \gamma_{(x_i)}(s) = f(x_i)$ for all $s \geq 0$. Furthermore, as in the proof of Theorem 3.1, $\gamma_{(x_i)}(s)$ traces out a straight line with slope (x horizontal, t vertical) $m_i \overset{\text{def}}{=} \frac{1}{f(x_i)}$. Since $\frac{1}{m_1} < \frac{1}{m_0}$, it is easy to check that $\gamma_{(x_0)}$ *intersects* $\gamma_{(x_1)}$ at the spacetime point $(t,x) = \left(\frac{x_1 - x_0}{\frac{1}{m_0} - \frac{1}{m_1}}, \frac{m_0 x_0 - m_1 x_1}{m_0 - m_1}\right)$. Thus, by Proposition 2.0.2 $u(t,x) = f(x_0)$ and $u(t,x) = f(x_1)$, which is a contradiction.

On the other hand, if $f'(p) \geq 0$ for all p, then for all $t_0 \geq 0$ and all x_0, the equation

$$x_0 = p + f(p)t_0 \tag{4.0.20}$$

has a unique solution $p = p_0(t_0, x_0)$ that depends on (t_0, x_0) in a C^1 fashion. This fact follows from e.g. the implicit function theorem since $\partial_p(p + f(p)t_0) = 1 + f'(p)t_0 > 0$ (i.e., the right-hand side of (4.0.20) is strictly increasing in p). Therefore, by Theorem 3.1 $u(t_0, x_0) = f \circ p_0(t_0, x_0)$, and $u \in C^1([0,\infty) \times \mathbb{R})$.

□

Exercise 4.0.1. Work through the details to to show that $\gamma_{(x_0)}$ intersects $\gamma_{(x_1)}$ at $(t,x) = \left(\frac{x_1 - x_0}{\frac{1}{m_0} - \frac{1}{m_1}}, \frac{m_0 x_0 - m_1 x_1}{m_0 - m_1}\right)$.

Exercise 4.0.2. Find a reference and review the implicit function theorem.

18.152 Introduction to Partial Differential Equations.
Fall 2011