Real analysis, Problem set 2

The first part of the pset is processing the Korn/Schauder regularity theory. Your main task is to carefully prove the boundary version of Korn's inequality. This is the hardest estimate in the theory, and it involves many of the tools.

1. Recall that $T_\epsilon f(x) := \int_{|y|>\epsilon} f(x-y)K(y)dy$, where K is a second derivative of the fundamental solution of the Laplacian, $K = \partial_i \partial_j \Gamma$. Recall that H is the upper half space $\{x | x_n > 0\}$.

Suppose that f is a compactly supported function, and that f restricted to the upper half-space H is in C^α, and that f restricted to the lower half-space H_- is in C^α. We allow f to be discontinuous across the boundary $\partial H = \{x | x_n = 0\}$.

The goal is to prove the following estimate. If $x, \bar{x} \in H$, and if $\epsilon < \min(x_n, \bar{x}_n)$, then

$$|T_\epsilon f(x) - T_\epsilon f(\bar{x})| \le C(n, \alpha)|x - \bar{x}|^\alpha \left([f]_{C^\alpha(H)} + [f]_{C^\alpha(H_-)} \right).$$

Here are some properties of K that we know and have been using, that you may find useful in the argument:

- $|K(y)| \le C_n |y|^{-n}$.
- $|\partial K(y)| \le C_n |y|^{-(n+1)}$.
- For any radius $r > 0$, $\int_{S_r} K(y) darea(y) = 0$.
- K is even: $K(y) = K(-y)$.
- K is homogeneous of degree $-n$. This means that for any constant $\lambda > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$, $K(\lambda x) = \lambda^{-n} K(x)$.

To be clear about the notation, recall that

$$[f]_{C^\alpha(H)} := \sup_{x,y \in H} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

This bound is a little tricky, and I want to highlight two special cases that are easier. You can do these two cases (for full credit) or the general case (for extra credit).

Case 1. $K = \partial_i \partial_j \Gamma$ with $i \neq j$. In this case, K enjoys an extra cancellation which makes the estimate easier:

$$K(y_1, ..., y_{i-1}, -y_i, y_{i+1}, ..., y_n) = -K(y_1, ..., y_{i-1}, y_i, y_{i+1}, ..., y_n).$$

Case 2. The case $x_n = \bar{x}_n$ is also a little easier.

(These two cases don't require the assumption that $\epsilon < \min(x_n, \bar{x}_n)$. For the general case, I believe this assumption is necessary, and it helps to use the homogeneity of K. Thanks to last year's class for clarifying these issues.)

2. Outline of the proof of global Schauder. Read through this outline of the proof of the global Schauder inequality. Most of the steps are the same as for the interior Schauder inequality. I recommend trying to fill in each step on your own. Then, if you get stuck, you can look at your notes about the original Schauder proof (and/or talk with classmates or come to office hours). I don't want to give you too much writing busywork, so the assignment is to pick any one of these six steps and fill in the details.

I'm also putting this .tex file on the website in case you want to make use of anything in your writeup.

a. Suppose that $u \in C^{2,\alpha}(\bar{H})$ with compact support and $u = 0$ on ∂H. Let $f = \triangle u$, defined on H, and extend f to H_- by $f(x_1, ..., x_{n-1}, x_n) = -f(x_1, ..., x_{n-1}, -x_n)$.

Proposition 1. *Under these hypotheses, for all $x \in H$,*

$$u(x) = \int_{\mathbb{R}^n} f(x-y)\Gamma(y)dy,$$

$$\partial_i \partial_j u(x) = \lim_{\epsilon \to 0} \int_{|y|>\epsilon} f(x-y)\partial_i\partial_j\Gamma(y)dy + (1/n)\delta_{ij} f(x).$$

b. Applying your estimate from problem 2, we get a boundary version of Korn's inequality:

Theorem 2. *Suppose that $u \in C^{2,\alpha}(\bar{H})$ with compact support and $u = 0$ on ∂H. Then*

$$[\partial_i \partial_j u]_{C^\alpha(H)} \le C(n, \alpha)[\triangle u]_{C^\alpha(H)}.$$

c. Now we can vary the coefficients slightly by using the rearrangement trick. We also add first-order terms to the operator L. These terms don't make the proof much more difficult, and they are useful in part f below.

Proposition 3. *For any dimension n and $0 < \alpha < 1$, there is a small $\epsilon(n, \alpha) > 0$ so that the following holds. Suppose that $u \in C^{2,\alpha}(\bar{H})$ with compact support in B_1 and with $u = 0$ on ∂H. Suppose that $Lu = \sum_{ij} a_{ij}\partial_i\partial_j u + \sum_i b_i \partial_i u$, where $|a_{ij} - \delta_{ij}| < \epsilon(n, \alpha)$ and $\|a_{ij}\|_{C^\alpha}, \|b_i\|_{C^\alpha} \le B$. Then:*

$$\|u\|_{C^{2,\alpha}(\bar{H})} \le C(n, B, \alpha)(\|Lu\|_{C^\alpha(\bar{H})} + \|u\|_{C^2(\bar{H})}).$$

By changing coordinates, we can also replace δ_{ij} by a positive definite matrix A_{ij} with $0 < \lambda \le A_{ij} \le \Lambda$. The ϵ then becomes $\epsilon(n, \alpha, \lambda, \Lambda)$ and the constant C in the final inequality becomes $C(n, B, \alpha, \lambda, \Lambda)$.

d. Next, we can allow general Holder coefficients by localizing with cutoff functions.

Proposition 4. *Suppose that $u \in C^{2,\alpha}(\bar{H} \cap B_1)$ with $u = 0$ on ∂H. Suppose that $Lu = \sum_{ij} a_{ij}\partial_i\partial_j u + \sum_i b_i \partial_i u$, where $0 < \lambda \le a_{ij} \le \Lambda$, and $\|a_{ij}\|_{C^\alpha}, \|b_i\|_{C^\alpha} \le B$. Then:*

$$\|u\|_{C^{2,\alpha}(\bar{H} \cap B_{1/2})} \le C(n, B, \alpha, \lambda, \Lambda)(\|Lu\|_{C^\alpha(\bar{H} \cap B_1)} + \|u\|_{C^2(\bar{H} \cap B_1)}).$$

e. A maximum principle for solutions of $Lu = f$. We know that if $Lu \ge 0$, then u obeys the maximum principle. If $|Lu|$ is small, then u obeys a version of the maximum principle with an error term. Prove this proposition:

Proposition 5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary. Suppose that $Lu = \sum_{ij} a_{ij}\partial_i\partial_j u + \sum_i b_i \partial_i u$, where $0 < \lambda \le a_{ij} \le \Lambda$ and $a_{ij}, b_i \in C^0(\Omega)$. If $u \in C^2(\bar{\Omega})$, prove that*

$$\max_{\bar{\Omega}} u \le \max_{\partial \Omega} u + C(\lambda, \Lambda, \Omega)\|Lu\|_{C^0(\Omega)}.$$

f. Finally, we come to the global Schauder inequality.

Theorem 6. *(Schauder) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary. Suppose that $Lu = \sum_{ij} a_{ij}\partial_i\partial_j u + \sum_i b_i \partial_i u$, where $0 < \lambda \leq a_{ij} \leq \Lambda$, and $\|a_{ij}\|_{C^\alpha}, \|b_i\|_{C^\alpha} \leq B$. Suppose that $u = \phi$ on $\partial\Omega$. Then:*

$$\|u\|_{C^{2,\alpha}(\bar{\Omega})} \leq C(n, B, \alpha, \lambda, \Lambda, \Omega) \left(\|Lu\|_{C^\alpha(\bar{\Omega})} + \|\phi\|_{C^{2,\alpha}(\bar{\Omega})} \right).$$

Using all the material above, prove the global Schauder inequality. To handle the parts near the boundary, you need to do a coordinate change that straightens the boundary, so that you can apply the results above. When we change the coordinates, we get first order terms even if the original operator only had second order terms, and this was the motivation to include these terms throughout.

The second (shorter) part of the problem set is to process the proof of the Sobolev inequality.

3. Suppose that u is a compactly supported smooth function on $\mathbb{R}^3$ whose derivatives obey the following L^p estimates:
$\|\partial_x u\|_{3/2} \leq 1; \|\partial_y u\|_{3/2} \leq 1; \|\partial_z u\|_1 \leq 1.$
For what p can you bound $\|u\|_p$?

Extra credit. Suppose that u is a compactly supported smooth function on $\mathbb{R}^2$ whose derivatives obey the following L^p estimates.
$\|\partial_x u\|_3 \leq 1; \|\partial_y u\|_4 \leq 1.$
For what α, if any, can you bound the Holder norm $[u]_\alpha$?

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