18.156 Lecture Notes

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Let us first recall what we did last time. Last time, we considered $u \in C^{2,\alpha}(B_1)$ and defined

$$Nu = (\Delta u - u^3, u|_{\partial B_1}),$$

where $N : C^{2,\alpha}(\overline{B_1}) \to C^\alpha(\overline{B_1}) \oplus C^{2,\alpha}(\partial B_1)$. We'll call these spaces X and Y, so $N : X \to Y$. We'll also denote B_1 as B.

Then, N is a C^1 map,

$$dN_u(v) = (\Delta v - 3u^2 v, v|_{\partial B})$$

is an isomorphism $X \to Y$ for all $u \in X$. And as a corollary of the inverse function theorem, if $F : X \to Y$ is C^1 and dF_x is an isomorphism, then the image of F contains a neighborhood of $F(x)$.

Proposition 1. *If $u \in C^2(\overline{B})$, and $Nu = (f, \varphi)$, then*

(i) $\|u\|_{C^0} \le \|\varphi\|_{C^0} + \|f\|_{C^0}$

(ii) $\|u\|_{C^{2,\alpha}(B)} \le g(\|f\|_{C^\alpha} + \|\varphi\|_{C^{2,\alpha}})$

An idea to prove (ii) is to first use global Schauder to get that

$$\begin{aligned}\|u\|_{C^{2,\alpha}(\overline{B})} &\lesssim \|\Delta u\|_{C^\alpha(B)} + \|\varphi\|_{C^{2,\alpha}(\partial B)} \\ &\le \|u^3\|_{C^\alpha} + \|f\|_{C^\alpha} + \|\varphi\|_{C^{2,\alpha}}.\end{aligned}$$

Then, we might try to use that

$$\|u^3\|_{C^\alpha} \le \|u\|_{C^\alpha}^3 \le (\epsilon \|u\|_{C^{2,\alpha}} + C_\epsilon \|u\|_{C^0})^3$$

and rearrange. But we have an exponent of 3, so this doesn't quite work. Instead, we take the inequality $\|fg\|_{C^\alpha} \le \|f\|_{C^0}\|g\|_{C^\alpha} + \|f\|_{C^\alpha}\|g\|_{C^0}$ and we have

$$\|u^3\|_{C^\alpha} \lesssim \|u\|_{C^0}^2 \|u\|_{C^\alpha} \le \|u\|_{C^0}^2 (\epsilon \|u\|_{C^{2,\alpha}} + C_\epsilon \|u\|_{C^0})$$

and now we can use rearrangement and the maximum principle to get the bounds that we want.

Theorem 2. *N is surjective from $X \to Y$.*

Proof. Given $(f, \varphi) \in Y$, define

$$SOL := \{t \in [0,1] : (tf, t\varphi) \in N(C^{2,\alpha}(B))\}.$$

We want to show that $1 \in SOL$. We already know that $0 \in SOL$, so it will suffice to show that SOL is open and closed. SOL is open since if $Nu = (t_0 f, t_0 \varphi)$, that dN_u is an isomorphism gives us that $N(C^{2,\alpha}(B))$ contains a neighborhood of $(t_0 f, t_0 \varphi)$.

To show that SOL is closed, suppose that $t_j \in SOL$ and $t_j \to t_\infty$, and $Nu_j = (t_j f, t_j \varphi)$. By the proposition, $\|u_j\|_{C^{2,\alpha}} \leq C$ uniformly in B. By the Arzela-Ascoli theorem, $u_j \to u_\infty$ in C^2 for a subsequence. And $Nu_\infty = \lim Nu_j = (f, \varphi)$. But

$$\|u_\infty\|_{C^{2,\alpha}} \leq \limsup \|u_j\|_{C^{2,\alpha}} \leq C,$$

so the limit is in $C^{2,\alpha}$. (We notice here that this does not say that $u_j \to u_\infty$ in $C^{2,\alpha}$, but says that $u_j \to u_\infty$ in C^2 and the limit is in $C^{2,\alpha}$, which is good enough for our purposes. $\square$

Question: if $\Delta u = 0$ on B, $u = \varphi$ on ∂B, then is $\|u\|_{C^1(B)} \lesssim \|\varphi\|_{C^1(\partial B)}$?

Here's a proof idea that doesn't quite work. We know that $\Delta \partial_i u = \partial_i \Delta u = 0$, so $\partial_i u$ obeys the maximal principle. We want to say now that

$$\|\partial_i u\|_{C^0} \leq \|\partial_i \varphi\|_{C^0} \leq \|\varphi\|_{C^1(\partial B)},$$

but the first inequality does not hold since φ does not have derivatives in as many directions as u does (it is missing the directions normal to ∂B). This idea of bounding the derivatives in the normal direction will be important later on.

Next examples:

(i) $\Delta u - |\nabla u|^2 = 0$: this has good global regularity and we can solve the Dirichlet problem.

(ii) $\Delta u - |\nabla u|^4 = 0$: this has no global regularity and we can't solve the Dirichlet problem.

Let us look at why the second case is bad. Take $n = 1$. Then, we are looking for solutions to

$$u'' - (u')^4 = 0.$$

If we take $w = u'$, then we want to solve $w' = w^4$. So $w^{-4} w' = 1$. But $(w^{-3})' = -3w^{-4}w' = -3$. From this, we get that $w(x)^{-3} = w(0)^{-3} - 3x$ and we have tat

$$w(x) = (w(0)^{-3} - 3x)^{-1/3}.$$

Now suppose that we want to solve $u(0) = 0$ and $u(1/3) = b$. For $0 \leq b < H$, this is solvable but for $b > H$, this is not solvable. We notice that if $b \to H$, then then the norm of the boundary data (the maximum of the values of the two points) is uniformly bounded, but $|u'(1/3)| \to \infty$, and this is what causes our problem.

Key Estimate: If $u \in C^2(\overline{\Omega})$, $\Delta u - |\nabla u|^2 = 0$, $u = \varphi$ on $\partial\Omega$, then

$$\|\partial_{nor} u\|_{C^0(\partial\Omega)} \leq C(\Omega)\|\varphi\|_{C^2(\partial\Omega)}.$$

(Note: this also gives us that $\|\partial u\|_{C^0(\partial\Omega)} \leq C(\Omega)\|\varphi\|_{C^2(\partial\Omega)}$.)

Proof Sketch: We want to construct $B : N \to \mathbb{R}$ such that

(i) $B(x_0) = u(x_0)$

(ii) $B \geq u$ on ∂N

(iii) $\Delta B - |\nabla B|^2 < 0$

(ii) and (iii) together will imply that $B \geq u$ on N. Then, $\partial_{nor} u(x_0) \leq \partial_{nor} B(x_0)$.

Proposition 3 (Comparison Principle)**.** *If*

$$Qu = \sum_{i,j} a_{ij}(\nabla u)\partial_i\partial_j u + b(\nabla u)$$

is a quasilinear elliptic PDE, where a_{ij} *are positive definite and* $a, b \in C^1$ *of* ∇u*, then if* $u, w \in C^2(\overline{\Omega})$*,* $u \leq w$ *on* $\partial\Omega$*,* $Qu \geq Qw$ *on* Ω*, then* $u \leq w$ *on* Ω

Proof of strict case. We want to show that $u - w \leq 0$ on Ω given that $u - w \leq 0$ on $\partial\Omega$ and $Q(u - w) > 0$. Suppose x_0 is an interior maximum. Then, $\nabla u(x_0) = \nabla w(x_0) = v_0$. Then,

$$\sum_{i,j} a_{ij}(v_0)\partial_i\partial_j(u - w)(x_0) > 0,$$

but this is impossible at a local maximum. □

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