18.156 Differential Analysis II
Lectures 22-24

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22.1 Precursor to Schrödinger Equation

Suppose we are trying to find $u(x,t)$ with $u(x,0) = f(x)$ satisfying one of the following three PDEs:

A: $\partial_t u = \partial_x^4 u$

B: $\partial_t u = -\partial_x^4 u$

C: $\partial_t u = i\partial_x^4 u$

We can take then take the Fourier transform of each of these, corresponding to

A': $\partial_t \widehat{u}(w,t) = (2\pi i w)^4 \widehat{u}(w,t)$

B': $\partial_t \widehat{u}(w,t) = -(2\pi i w)^4 \widehat{u}(w,t)$

C': $\partial_t \widehat{u}(w,t) = i(2\pi i w)^4 \widehat{u}(w,t)$

Each of these can now be formally solved in this frequency domain, since these are just separable differential equations (and the initial data f corresponds to initial data $\widehat{f}$). However, such solutions might or might not correspond to legitimate solutions u, since if the formal solution for $\widehat{u}$ is not well enough behaved in w, then one cannot take the Fourier transform to get a corresponding solution u. We analyze each case separately.

A: Solving yields $\widehat{u}(w,t) = \exp(Cw^4 t)\widehat{f}(w)$. For almost all initial data, this solution goes bananas for $t > 0$, because $\exp(Cw^4 t)$ grows too fast. Thus, we can't possibly take a Fourier transform to find a legitimate solution $u(x,t)$. For example, if $f = e^{-\pi x^2}$, which decays quite nicely, then $\widehat{f} = e^{-\pi w^2}$, which although it still decays nicely, is taken over by $\exp(Cw^4 t)$ for any $t > 0$. Of course, if $\widehat{f} \in C_c^\infty$, then we get a solution.

B: Solving yields $\widehat{u}(w,t) = \exp(-Cw^4 t)\widehat{f}(w)$. For $t > 0$, for most $\widehat{f}$, u will become immediately quite smooth. If $f \in L^1$, then $\widehat{f}$ is bounded, and so $\widehat{u}(w,t)$ decays rapidly, which means that taking the Fourier transform, $u(x,t)$ is C^∞. In fact, we can say a bit more.

Proposition 1. $\|\partial_x^k u(x,1)\|_{L^\infty} \le C_k \|f\|_{L^1}$

Proof. We have

$$\begin{aligned} |\partial_x^k u(x,1)| &\le \|\widehat{\partial_x^k u}(w,1)\|_{L^1_w} \\ &= \|(2\pi i w)^k e^{-iw^4} \widehat{f}(w)\|_{L^1} \\ &\le \|(2\pi i w)^k e^{-iw^4}\|_{L^1} \|\widehat{f}\|_{L^\infty} \\ &\le C_k \|f\|_{L^1} \end{aligned}$$

□

Even further, one can use either a scaling argument (since if $u(x,t)$ is a solution to the differential equation, so is $u(\lambda x, \lambda^4 t)$) or a direct argument (presented below) to prove the following proposition:

Proposition 2. $\|u(x,t)\|_{L^\infty} \lesssim t^{-1/4}\|f\|_{L^1}$

Proof.

$$\begin{aligned} |u(x,t)| &\leq \|\widehat{u}(w,t)\|_{L^1_w} \\ &= \|e^{-Cw^4 t}\widehat{f}(w)\|_{L^1} \\ &\leq \|e^{-Cw^4 t}\|_{L^1}\|\widehat{f}\|_{L^\infty} \\ &\leq t^{-1/4}C\|f\|_{L^1} \end{aligned}$$

□

C: Solving in the frequency domain yields (up to rescaling) $\widehat{u}(w,t) = e^{iw^4 t}\widehat{f}(w)$. Note that this means that if $\widehat{f}$ is Schwartz, then so is $\widehat{u}$, so in particular, if f is itself Schwartz, then so is $\widehat{f}$, hence $\widehat{u}$, and so we can take the Fourier transform and get a Schwartz function u satisfying the PDE. So at the very least, we can solve for f Schwartz, which is a decently large class.

Proposition 3. *$\|\partial_x^k u(x,t)\|_{L^2_x}$ is independent of t for each $k \geq 0$.*

Proof. By an application of Plancherel's Theorem, this is equal to $\|(2\pi i w)^k e^{iw^4 t}\widehat{f}(w)\|_{L^2_w}$, and so the exponential term has constant modulus 1 and does not affect this value. □

What this means is that studying C is much more subtle than studying A or B, since the previous proposition tells us that the 'total mass' in the solution of this equation is independent of t, so things will not just blow up or become more regular as t increases. If we want to study decay, we need to be careful. By scaling, if solutions to this PDE do decay, then it would need to do so in a way resembling the following:

$$\|u(x,t)\|_{L^\infty_x} \lesssim t^{-1/4}\|f\|_{L^1}.$$

The main question is, does this hold?

22.2 The Schrödinger Equation

Instead of PDE C from the previous subsection, we will focus on the Schrödinger equation, which as we will see immediately behaves quite similarly to the above.

The Schrödinger Equation: Find $u(x,t)$, $x \in \mathbb{R}^d$, $t \in \mathbb{R}$, satisfying the initial condition $u(x,0) = f(x)$ such that

$$\partial_t u = i\Delta u.$$

In the frequency domain, this amounts to solving the equation $\widehat{u}(w,t) = e^{i|w|^2 t}\widehat{f}(w)$, and issues of convergence are relatively similar to case C of the previous subsection. In fact, the same exact proves carry over to show:

Proposition 4. *If $f \in \mathcal{S}$, then we get a solution $u \in \mathcal{S}$.*

Proposition 5. *For any multi-index I, $\|\partial_x^I u(x,t)\|_{L^2_x}$ is independent of t.*

We wish to understand the decay of the Schrödinger equation. In particular, we will prove the following proposition:

Proposition 6. $\|u(x,t)\|_{L^\infty_x} \lesssim t^{-d/2}\|f\|_{L^1}$.

Remark 7. There is in fact a nice formula for the solution, given by $u(x,t) = t^{-d/2}e^{i|x|^2/4t} * f(x)$, so that the proposition becomes trivial. However, we want to have a more generalizable gadget for completing this proof.

Proof. We will complete the proof in the next lecture. For now, let us reduce the problem to a lemma.

First of all, note that

$$\begin{aligned} u(x,t) &= \int \widehat{u}(w,t)e^{iwx}dw \\ &= \lim_{\delta\to 0}\int e^{-\delta w^2}e^{i|w|^2 t}\widehat{f}(w)e^{iwx}dw \end{aligned}$$

Now if we set $S_{\delta,t}(x)$ equal to the inverse Fourier transform of $e^{-\delta|w|^2}e^{i|w|^2 t}$ with respect to w, then we see that

$$u(x,t) = \lim_{\delta\to 0} S_{\delta,t}(x) * f(x).$$

In particular, we included the term $e^{-\delta|w|^2}$ so that we could take this Fourier transform in the first place. Hence, we find that

$$\|u(x,t)\|_{L^\infty} \le \liminf \|S_{\delta,t}\|_{L^\infty}\|f\|_{L^1}.$$

It suffices therefore to prove that $\|S_{\delta,t}\|_{L^\infty} \lesssim t^{-d/2}$ uniformly as $\delta \to 0$. By scaling, we need only prove in fact that $\|S_{\delta,1}\|_{L^\infty}$ is uinformly bounded by a constant. But this integral breaks up into a product as follows:

$$\begin{aligned} |S_{\delta,1}(x)| &= \left|\int_{\mathbb{R}^d} e^{-\delta|w|^2}e^{i|w|^2}e^{iw\cdot x}\ dw\right| \\ &= \prod_{j=1}^{d}\left|\int_{\mathbb{R}} e^{-\delta w_j^2}e^{iw_j^2}e^{iw_jx_j}\ dw_j\right| \end{aligned}$$

So it suffices to prove the one-dimensional case, the second of the following two lemmas:

Lemma 1. $\left|\int_{\mathbb{R}} e^{-\delta w^2}e^{iw^2}\ dw\right| \lesssim 1$ *uniformly in* δ.

Lemma 2. $\left|\int_{\mathbb{R}} e^{-\delta w^2}e^{iw^2}e^{iwx}\ dw\right| \lesssim 1$ *uniformly in* δ, x.

We will prove these next time. Again, the second of these lemmas is what we actually need, but proving the first gives us a good indication for how we might prove the second. □

Lecture 23: 6 April 2015

23.1 Completion of proof

Proof Sketch of Lemma 1. Set $\eta = w^2$. Then changing variables accordingly, we see

$$\left|\int_0^\infty e^{-\delta w^2}e^{iw^2}dw\right| = \left|\int_0^\infty e^{-\delta\eta}\eta^{-1/2}e^{i\eta}d\eta\right|,$$

and this kind of looks similar to the integration estimates in the Gauss circle problem. A bounded distance away from 0, we can use the triangle inequality, while away from 0, we can integrate by parts. The main term of integration will be $\eta^{-3/2}e^{i\eta}$, which is integrable. □

Proof Sketch of Lemma 2. This is similar, but now e^{iw^2} is replaced with $e^{i(wx+w^2)}$. But $w^2 + wx$ is just a quadratic centered around $w_0 := -x/2$, and the same proof carries over where we change coordinates to integrate around this center instead. □

Remark 8. There is an alternative proof of these two. The oscillatory term gets faster and faster away from w_0, and so if we break the integral up into corresponding full periods in the real and imaginary parts, we will get alternating sums which are at most as large as the first term, which can be bounded by the triangle inequality.

23.2 Statement of Strichartz Theorem

In fact, we see that by the same method, we can study $|S_{\delta,t}(x)|$ still for all δ, x, but also for all t, not just $t \geq 0$. Say we fix $\delta = 1$. Of course, we know that the **mass** given by $\|S_1(t,x)\|_{L^2_x}$ remains constant in time. At time $t = 0$, $S_1(t,x)$ has about height 1 and width 1, while at time $|t| >> 1$, it is defocused with height about $t^{-d/2}$ and width about t. For negative t, we get **focusing data**, while for positive t, we get **defocusing data**, because if we solve the Schrödinger equation with this initial data, we just flow with t getting larger, and for starting at negative t, this collimates around $t = 0$ and then defocuses again. Qualitatively, the focusing and defocusing data are quite similar, so one expects it to be difficult to use purely qualitative estimates to learn about what happens as we flow in time.

Nonetheless, we might be interested in what kinds of estimates we can still obtain. To motivate this, consider the following question:

Question 1. For which p do we have the estimate

$$\|u(x,t)\|_{L^p_{x,t}} \lesssim \|f\|_{L^2} = \text{mass}^{1/2}?$$

Intimately related is the question of how large is the set

$$\{(x,t)\colon |u(x,t)| > t\}?$$

Remark 9. For the function S_1 we have been considering, we get such an estimate if and only if $p > \frac{2(d+1)}{d}$, which we will see later.

So what about solutions to the Schrödinger equation? Note that if u solves the Schrödinger equation, then $u_\lambda(x,t) := u(x/\lambda, t/\lambda^2)$ also solves the Schrödinger equation with initial data $f_\lambda(x) = f(x/\lambda)$, and we have

$$\|u_\lambda\|_{L^p_{x,t}} \sim \lambda^{(d+2)/p}\|u\|_{L^p_{x,t}}, \qquad \|f_\lambda\|_{L^2} \sim \lambda^{d/2}\|f\|_{L^2}.$$

Hence, the only possible p which could possibly answer our question is when $p = 2(d+2)/d$. In fact, this equation is satisfied uniformly for all S_δ uniformly in δ if and only if $p = 2(d+2)/d$. Of course, it is worth noting that $S_{\delta,t}(x)$ is by definition the solution to Schrödinger equation with initial data given by a Gaussian (in particular the inverse Fourier transform of $e^{-\delta w^2}$). So one might expect the following theorem:

Theorem 10 (Strichartz, 1970's)**.** *Set $p = 2(d+2)/d$. Then the estimate $\|u(x,t)\|_{L^p_{x,t}} \lesssim \|u(\bullet,0)\|_{L^2_x}$ holds for all solutions to the Schrödinger equation.*

The proof of this theorem will come later, but even here, we stumble upon a general sort of analytic question:

Question 2. Let T be a given linear operator. Then find all p, q such that there is an inequality of the form

$$\|Tf\|_{L^p} \lesssim \|f\|_{L^q}.$$

Example 11. Just something to think about: Consider $g(x) := e^{ix^2}(1+|x|)^{-1/2}$, and let $Tf := f * g$. Then for what (p,q) do we get an inequality as above.

This is of course a difficult question, and currently, there is no universal way to solve this question. However, there are many tools which can be used to attack it. We will see how interpolation is one extremely useful tool, allowing us to take two given estimates of this form and yield an interval-worth of these estimates.

23.3 Interpolation

Interpolation of L^p spaces comes in two flavors, both from the first half of the 20th century. The first is due to Riesz-Thorin, while the second is due to Marcinkiewicz. We will follow the Marcinkiewicz approach instead, although see Remark 13. We begin with a special case of interpolation.

Theorem 12 (Interpolation, Special Case)**.** *If T is a linear operator, $1 \leq p < r < q \leq \infty$, and $\|Tf\|_{L^p} \leq \|f\|_{L^p}$ for all $f \in L^p$ and $\|Tf\|_{L^q} \leq \|f\|_{L^q}$ for all $f \in L^q$, then for all $f \in L^r$, we have*

$$\|Tf\|_{L^r} \leq C(p,q,r)\|f\|_{L^r}.$$

Remark 13. In this special case, Riesz-Thorin actually gives $C(p,q,r) = 1$, which is optimal. What the Riesz-Thorin approach gives in terms of better bounds, the Marcinkiewicz approach makes up for in slightly more general hypotheses, to be stated later, except for some fairly uninteresting cases which Marcinkiewicz doesn't cover but Riesz-Thorin does. We shall not concern ourselves with these matters from here on.

We won't quite finish the proof today, but we'll still do a little bit of work. The idea of the proof of interpolation is to look at f and Tf at different height scales. Define

$$S_k f := \{x\colon 2^k \leq |f(x)| < 2^{k+1}\}, \qquad f_k = f \cdot \chi_{S_k f}.$$

Then we have $f = \sum_k f_k$, and we get $\|f\|_{L^p}^p \sim \sum_k |S_k(f)| 2^{pk}$. The idea now is to use estimates on $|S_\ell(Tf_k)|$ and glue them together carefully to arrive at a proof. We shall work through this over the remainder of this lecture and into next lecture. We begin with the estimate on $|S_\ell(Tf_k)|$.

Proposition 14. *If $\|Tf\|_p \leq \|f\|_p$ for all $f \in L^p$, then $|S_\ell(Tf_k)| \leq 2^p \cdot 2^{(k-\ell)p} \cdot |S_k(f)|$.*

Proof. We have

$$|S_\ell(Tf_k)| \cdot 2^{\ell p} \leq \|Tf_k\|_p^p \leq \|f_k\|_p^p \leq 2^p \cdot 2^{kp} |S_k(f)|.$$

□

Hence, the proposition above gives us bounds on $|S_\ell(Tf_k)|$ with respect to both p and q. Say $p < q$. Then we see that if $\ell > k$, then the q-bound is better, and vice versa if $\ell < k$. This allows us to prove the following corollary:

Corollary 15. If $\|Tf\|_p \leq \|f\|_p$ for all $f \in L^p$ and $\|Tf\|_q \leq \|f\|_q$ for all $f \in L^q$ with $p < q$, then there is a constant $\epsilon(p,q,r)$ such that for $p < r < q$,

$$\|(Tf_k)_\ell\|_{L^r} \lesssim 2^{-\epsilon|\ell-k|} \|f_k\|_{L^r}.$$

Proof. Suppose first that $\ell \geq k$. Then using the q-bounds, we find

$$\|(Tf_k)_\ell\|_r^r \lesssim 2^{r\ell} |S_\ell(Tf_k)| \leq 2^{r\ell} \cdot 2^q \cdot 2^{-(\ell-k)q} |S_k(f)| = \|f_k\|_r^r 2^{r\ell} 2^{-rk} 2^q 2^{-(\ell-k)q}$$

and so

$$\|(Tf_k)_\ell\|_r \lesssim 2^{(\ell-k)(1-q/r)} 2^{q/r} \|f_k\|_r \leq 2^{(\ell-k)(1-q/r)} 2^{q/p}$$

and $1 - q/r < 0$. We have a similar bound when $k \geq \ell$, but instead we get our ϵ value is $p/r - 1$ (and the extra $2^{q/p}$ term disappears, though this doesn't matter). □

Lecture 24: 8 April 2015

24.1 Finishing the special case

Let's add some extra notation. Set $V_\lambda(f) := |\{x\colon |f(x)| > \lambda\}|$. Of course, $V_\lambda(f) \cdot \lambda^p \leq \|Tf\|_p^p$. Now, we will actually show that the theorem we stated last time is true in a slightly weaker setting (and in fact, we've already down some of the work).

Theorem 16 (Interpolation, Take II)**.** *Suppose T is sublinear (meaning $|T(f+g)(x)| \leq |Tf(x)| + |Tg(x)|$), and we have the so-called weak type estimates $V_\lambda(Tf) \cdot \lambda^p \leq \|f\|_p^p$ for all $f \in L^p$ and $V_\lambda(Tf) \leq \|f\|_q^q$ for all $f \in L^q$, say with $1 \leq p < r < q \leq \infty$. Then*

$$\|Tf\|_{L^r} \leq C(p,q,r) \|f\|_{L^r}.$$

Remark 17. If we had the weak type estimates, but with constants A_p and A_q, then our concluding estimate would have an extra factor of $A_p^\alpha A_q^{1-\alpha}$, with $\frac{1}{r} = \alpha \frac{1}{p} + (1-\alpha)\frac{1}{q}$. This is a simple scaling argument, so it suffices to prove for $A_p = A_q = 1$.

We have

Lemma 3. $V_\lambda(Tf_k) \leq (2^k/\lambda)^p |S_k(f)|$ *(and also for q).*

Proof. $V_\lambda(Tf_k) \cdot \lambda^p \leq \|f_k\|_p^p \sim |S_k(f)| 2^{kp}$. □

Lemma 4. $V_{2^\ell}(Tf_k) \cdot 2^{\ell r} \lesssim |S_k(f)| \cdot 2^{kr} \cdot 2^{-\epsilon|k-\ell|}$

Proof. Essentially done last time. □

Corollary 18. $\|Tf_k\|_{L^r} \lesssim \|f_k\|_{L^r}$

Proof. We have $Tf_k \leq \sum_\ell 2^\ell \chi_{V_{2^\ell}}(Tf_k)$, so taking L^r norms and using the previous lemma gives the result. □

Our goal is now to add the contributions from all of these height scales. The naïve approach would be to simply use the sublinearity of T to give

$$\|Tf\|_{L^r} \leq \sum_k \|Tf_k\|_{L^r} \lesssim \|f_k\|_{L^r}$$

At this point, we'd have trouble bounding by $\|f\|_{L^r}$. Instead, we want to use the info from our bounds on $|S_k(f)|$ to change the first inequality. The correct way to do this is with the following lemma, which is the tricky part of the argument.

Lemma 5. $V_{Tf}(2^\ell) \lesssim \sum_k |S_k(f)| \cdot 2^{kr} \cdot 2^{-\ell r} \cdot 2^{-\epsilon|k-\ell|}$

Proof. This would be easy if $|Tf(x)| \geq 2^\ell$ implied that there exists a k with $|Tf_k(x)| \geq 2^\ell$. Unfortunately, we don't have this, but we can be a little bit more clever. For each ℓ, choose weights w_k with $\sum_k w_k = 1$, which we will choose later. Then the sublinearity of T means that if $|Tf(x)| \geq 2^\ell$, then there is some k with $|Tf_k(x)| \geq w_k \cdot 2^\ell$. So we find that

$$V_{2^\ell}(Tf) \leq \sum_k V_{w_k \cdot 2^\ell}(Tf_k) \leq \sum_k |S_k(f)| \cdot 2^{kr} \cdot 2^{-\ell r} \cdot w_k^{-c} \cdot 2^{-\epsilon|k-\ell|}.$$

So if we choose $w_k \approx \frac{1}{(\ell-k)^2}$ with constant chosen so that $\sum w_k = 1$, then the w_k^{-c} term is dominated by the exponential decay of the $2^{-\epsilon|k-\ell|}$ term, and so we can replace ϵ by an arbitrarily smaller value and obtain the same result. □

The rest of the proof is now easy.

$$\begin{aligned} \|Tf\|_r^r &\lesssim \sum_\ell V_{2^\ell}(Tf) \cdot 2^{\ell r} \\ &\lesssim \sum_\ell \sum_k |S_k(f)| 2^{kr} 2^{-\epsilon|k-\ell|} \end{aligned}$$

Summing in ℓ, we see this is on the order of $\sum_k |S_k(f)| 2^{kr}$, which is itself about $\|f\|_r^r$.

24.2 Statement of the General Case

The following is essentially the content of a problem on the 4th PSet, and is again discussed in the April 15 notes.

Theorem 19 (Marcinkiewicz Interpolation)**.** *Suppose $1 \leq p_i \leq q_i \leq \infty$ for $i = 0, 1$ with $q_0 \neq q_1$. If T sublinear and for all f in the appropriate spaces we have the weak type estimates that for all λ,*

$$(V_\lambda(Tf) \cdot \lambda^{q_i})^{1/q_i} \leq A_i \|f\|_{p_i}$$

then for each $0 < \theta < 1$, setting $\frac{1}{p_\theta} = \theta \cdot \frac{1}{p_0} + (1-\theta) \cdot \frac{1}{p_1}$ and similarly for q_θ, we have strong type estimates of the form

$$\|Tf\|_{q_\theta} \lesssim_\theta A_0^{1-\theta} A_1^\theta \|f\|_{p_\theta}.$$

In other words, the space of estimates of the form $\|Tf\|_q \lesssim \|f\|_p$ is convex in the $(1/p, 1/q)$-domain, and in order to interpolate, one only needs weak estimates at the boundary.

Question 3. Here's a challenge. Suppose we are given (p_0, q_0), (p_1, q_1) as in the hypotheses of the theorem. Is there a sublinear operator T such that $\|Tf\|_q \lesssim \|f\|_p$ if and only if (p, q) lie in the interpolating interval between these two endpoints? How about if we choose any convex subset of the $(1/p, 1/q)$-domain?

Remark 20. If T is the Fourier transform on $\mathbb{R}^d$, then we have strong estimates on L^2 by Plancherel (in fact equality) and L^∞ by the triangle inequality. Hence, we get strong estimates by interpolation of the form

$$\|\widehat{f}\|_{L^q} \lesssim \|f\|_{L^p}$$

with $\frac{1}{p} + \frac{1}{q} = 1$. In fact, these are the only such estimates for the Fourier transform.