18.156 Lecture Notes

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Today, we'll start the proof of the Calderon Zygmund theorem, which we recall here:

Theorem 1. *If* $Tf = f * K$ *on* $\mathbb{R}^d$, $|K(x)| \lesssim |x|^{-d}, |\partial K(x)| \lesssim |x|^{-d-1}, \int_{S_r} K(x) = 0$ *for all* r, *then* $\|Tf\|_p \lesssim \|f\|_p, 1 < p < \infty$.

This proof will be split into four parts, as discussed last class.

Part I: L^2 **bound.** By Plancherel, we have that

$$\|f * K\|_2 = \|\hat{f} \cdot \hat{K}\|_2 \leq \|\hat{K}\|_\infty \cdot \|\hat{f}\|_2 = \|\hat{K}\|_\infty \cdot \|f\|_2.$$

So it suffices to bound $\|\hat{K}\|_\infty$. We have that

$$\begin{aligned} |\hat{K}(\omega)| &= \left| \int K(x) e^{-i\omega x}\, dx \right| \\ &\leq \sum_{j \in \mathbb{Z}} \left| \int_{2^{j-1} \leq |x| \leq 2^j} K(x) e^{-i\omega x}\, dx \right| \\ &=: \sum_{j \in \mathbb{Z}} I_j. \end{aligned}$$

We'll also let

$$A_j := \{x : 2^{j-1} \leq |x| \leq 2^j\}.$$

For small j, those such that $|\omega \cdot 2^j| \leq 1$, we have from $\int_{S_r} K(x) = 0$ that

$$\begin{aligned} |I_j| &= \left| \int_{A_j} K(x)(e^{-i\omega x} - 1)\, dx \right| \\ &\leq |\omega \cdot 2^j| \int_{A_j} |K(x)| \\ &\sim |\omega \cdot 2^j| (2^j)^d \cdot (2^j)^{-d} \\ &\sim |\omega \cdot 2^j|. \end{aligned}$$

But now, $\sum_{|\omega w^j| \leq 1} I_j$ is the sum of exponentially decreasing terms, and is therefore $\lesssim 1$. We also have to worry about what happens for large j. For j such that $|\omega \cdot 2^j| > 1$, we can choose

$\ell \in \{1, 2, \ldots, d\}$ such that $|\omega_\ell| \gtrsim |\omega|$ and integrate by parts to get that

$$\begin{aligned}
|I_j| &= \left| \int_{A_j} K(x) e^{-i\omega x}\, dx \right| \\
&= \left| \int_{A_j} \partial_\ell K \cdot \frac{1}{i\omega_\ell} e^{-i\omega x}\, dx + \int_{\partial A_j} K e^{-i\omega x} \cdot \frac{1}{i\omega_\ell}\, dx \right| \\
&\le \int_{A_j} |\partial K| \cdot \frac{1}{|\omega|}\, dx + \int_{\partial A_j} |K| \cdot \frac{1}{|\omega|} \\
&\lesssim |A_j| (2^j)^{-d-1} \cdot \frac{1}{|\omega|} + |\partial A_j| \cdot (2^j)^{-d} \cdot \frac{1}{\omega} \\
&\sim \frac{1}{|2^j \omega|}.
\end{aligned}$$

Again, $\sum_{|\omega 2^j| > 1} I_j$ is bounded by an exponentially decaying series, and so this sum and therefore the whole sum $\lesssim 1$. This gives us the L^2 bound that we wanted.

We note here that sometimes in the statement of Calderon Zygmund, the L^2 bound $\|Tf\|_2 \lesssim \|f\|_2$ is taken to be a condition instead of $\int_{S_r} K(x) = 0$.

Part II: Weak L^1 bound. We want to prove the statement

$$V_{Tf}(\lambda) \lesssim \|f\|_1 \lambda^{-1}. \tag{1}$$

We will do this by breaking up the function f into a small part and a "balanced part". Let us first show that a weak L^1 bound holds for "small" and "balanced" functions. We'll start with small functions.

Lemma 2. *If $\|f\|_\infty \le 10\lambda$, then (1) holds.*

Proof. This follows from the L^2 estimate.

$$V_{Tf}(\lambda) \le \|Tf\|_2^2 \cdot \lambda^{-2} \lesssim \|f\|_2^2 \cdot \lambda^{-2} \lesssim \|f\|_1 \cdot \lambda^{-1}.$$

□

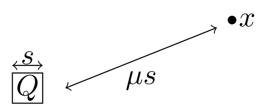
For example, if we had the function $f = H \cdot \chi_{B_r}$ for $\lambda \ll H$. Then, we would have that

$$|Tf(x)| \lesssim f * |x|^{-d} =: g.$$

And $\lambda = H \cdot r^d \cdot R^{-d}$ so $R^d \cdot \lambda \sim H \cdot r^d \sim \|f\|_{L^1}$, and this bound makes sense.

Here's another example where we couldn't employ this reasoning. Let $f = \sum_j \chi_{B_j}$ where $B_j = B(x_j, r)$ and x_j are spaces with spacing s in a large finite grid. Then, again, we would have that $|Tf| \lesssim |f * |x|^{-d}|$, but it is an exercise to check that the right hand side is too big to get a bound of the type that we want. Instead, we have to use that $|Tf| \ll |f * |x|^{-d}|$ by cancellation.

Lemma 3. *If $b(x)$ is "balanced for λ", supp $b \subset$ cube Q, $⨍_Q |b| = \lambda, \int_Q b = 0$, then $|Tb(x)| \leq \lambda \cdot \mu^{-d-1}$. Here, μs is the distance from x to Q and $\mu \geq 2$.*



Proof. Note that

$$|Tb(x)| \leq \int_Q |b| \cdot |K(x-y|\,dy \sim (\mu \cdot s)^{-d} \int_Q |b| \sim \lambda \cdot \mu^{-d},$$

but we can do better than that. If y_0 is the center of Q, then have that

$$\begin{aligned} |Tb(x) &= \left| \int_Q b(y) K(x-y)\,dy \right| \\ &= \left| \int_Q b(y)(K(x-y) - K(x-y_0))\,dy \right|. \end{aligned}$$

Now, since $K(x-y) - K(x-y_0) \lesssim s \cdot \max_{y \in Q} |\partial K(x-y)| \lesssim s \cdot (\mu s)^{-d-1}$, we have that

$$|Tb(x)| \lesssim s \cdot (\mu s)^{-d-1} \int |b(y)| \sim \mu^{-d-1} \cdot \lambda.$$

□

Lemma 4. *If $b = \sum b_j$, b_j balanced functions for λ, and each function b_j is supported on Q_j disjoint sets, then $V_{Tb}(\lambda) \lesssim \|b\|_1 \cdot \lambda^{-1}$.*

Proof. We have that $\|b\|_1 \sim \lambda \sum_j |Q_j|$. Let $U := \bigcup_j 2Q_j$. Then, $|U| \lesssim \|b\|_1 \cdot \lambda^{-1}$. So it suffices to check that $\|Tb\|_{L^1(\mathbb{R}^d \setminus U)} \lesssim \|b\|_1$, and for this it suffices to check that $\|Tb_j\|_{L^1(\mathbb{R}^d \setminus 2Q_j)} \lesssim \|b_j\|_1$, since then we would have that

$$\|Tb\|_{L^1(\mathbb{R}^d \setminus U)} \leq \sum_j \|Tb_j\|_{L^1(\mathbb{R}^d \setminus U)} \leq \sum_j \|Tb_j\|_{L^1(\mathbb{R}^d \setminus 2Q_j)} \lesssim \sum_j \|b_j\|_1 = \|f\|_1,$$

since the b_j have disjoint supports. But that $\|Tb_j\|_{L^1(\mathbb{R}^d \setminus 2Q_j)} \lesssim \|b_j\|_1$ follows from integrating the last lemma. □

Our next step will be to decompose functions into balanced and small parts so we can use the above results.

Lemma 5 (Calderon-Zygmund Decomposition Lemma)**.** *For all $f \in C_c^0$, $\lambda > 0$, we can decompose $f = b + s$ where $\|b\|_1 + \|s\|_1 \lesssim \|f\|_1$, $\|s\|_{L_\infty} \leq \lambda$, $b = \sum b_j$ where b_j is balanced for λ and supported on disjoint Q_j, where $⨍_{Q_j} b_j \lesssim ⨍_{Q_j} f \lesssim \lambda$.*

We'll prove this lemma next time, but we can first show that this lemma will imply part II of the proof of CZ. Given this lemma, we would have that

$$V_{Tf}(2\lambda) \leq V_{Ts}(\lambda) + V_{Tb}(\lambda) \lesssim \|s\|_1 \lambda^{-1} + \|b\|_1 \lambda^{-1} \lesssim \|f\|_1 (2\lambda)^{-1}.$$

Let's conclude today with an example of how we might split a function f into a small and a balanced part. Let

$$f = \sum_j \chi_{B_j}$$

where $B_j = B(x_j, 1)$ and x_j are in a grid with spacing $\gg 1$, $s^{-d} \leq \lambda \ll 1$. Then, we could choose cubes Q_j of width s centered at the x_j such that $⨍_{Q_j} |f| \sim \lambda$. Then, we could let $s = \sum_j \lambda \chi_{Q_j}$ and $b_j = \chi_{B_j} - \lambda \chi_{Q_j}$.