

4. MATRIX INTEGRALS

Let \mathfrak{h}_N be the space of Hermitian matrices of size N . The inner product on \mathfrak{h}_N is given by $(A, B) = \text{Tr}(AB)$. In this section we will consider integrals of the form

$$Z_N = \hbar^{-N^2/2} \int_{\mathfrak{h}_N} e^{-S(A)/\hbar} dA,$$

where the Lebesgue measure dA is normalized by the condition $\int e^{-\text{Tr}(A^2)/2} dA = 1$, and $S(A) = \text{Tr}(A^2)/2 - \sum_{m \geq 0} g_m \text{Tr}(A^m)/m$ is the action functional.³ We will be interested the behavior of the coefficients of the expansion of Z_N in g_i for large N . The study of this behavior will lead us to considering not simply Feynman graphs, but actually fat (or ribbon) graphs, which are in fact 2-dimensional surfaces. Thus, before we proceed further, we need to do some 2-dimensional combinatorial topology.

4.1. Fat graphs. Recall from the proof of Feynman's theorem that given a finite collection of flowers and a pairing σ on the set T of endpoints of their edges, we can obtain a graph Γ_σ by connecting (or gluing) the points which fall in the same pair.

Now, given an i -flower, let us inscribe it in a closed disk D (so that the ends of the edges are on the boundary) and take its small tubular neighborhood in D . This produces a region with piecewise smooth boundary. We will equip this region with an orientation, and call it a *fat i -valent flower*. The boundary of a fat i -valent flower has the form $P_1 Q_1 P_2 Q_2 \dots P_i Q_i P_1$, where P_i, Q_i are the angle points, the intervals $P_j Q_j$ are arcs on ∂D , and $Q_j P_{j+1}$ are (smooth) arcs lying inside D (see Fig. 10).

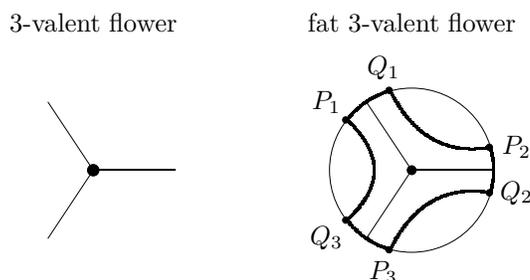


FIGURE 10

Now, given a collection of usual flowers and a pairing σ as above, we can consider the corresponding fat flowers, and glue them (respecting the orientation) along intervals $P_j Q_j$ according to σ . This will produce a compact oriented surface with boundary (the boundary is glued from intervals $P_j Q_{j+1}$).

We will denote this surface by $\tilde{\Gamma}_\sigma$, and call it *the fattening* of Γ with respect to σ . A fattening of a graph will be called a *fat (or ribbon) graph*. Thus, a fat graph is not just an oriented surface with boundary, but such a surface together with a partition of this surface into fat flowers.

Note that the same graph Γ can have many different fattenings, and in particular the genus g of the fattening is *not* determined by Γ (see Fig. 11).

4.2. Matrix integrals in large N limit and planar graphs. Let us now return to the study of the integral Z_N . By the proof of Feynman's theorem,

$$\ln Z_N = \sum_{\mathbf{n}} \left(\prod_i \frac{g_i^{n_i} \hbar^{n_i(i/2-1)}}{i^{n_i} n_i!} \right) \sum_{\sigma} F_{\sigma},$$

where the summation is taken over all pairings of $T = T(\mathbf{n})$ that produce a connected graph Γ_σ , and F_σ denotes the contraction of the tensors $\text{Tr}(A^i)$ using σ .

For a surface Σ with boundary, let $\nu(\Sigma)$ denote the number of connected components of the boundary.

Proposition 4.1. $F_\sigma = N^{\nu(\tilde{\Gamma}_\sigma)}$.

³Note that we divide by m and not by $m!$. We will see below why such normalization will be more convenient.

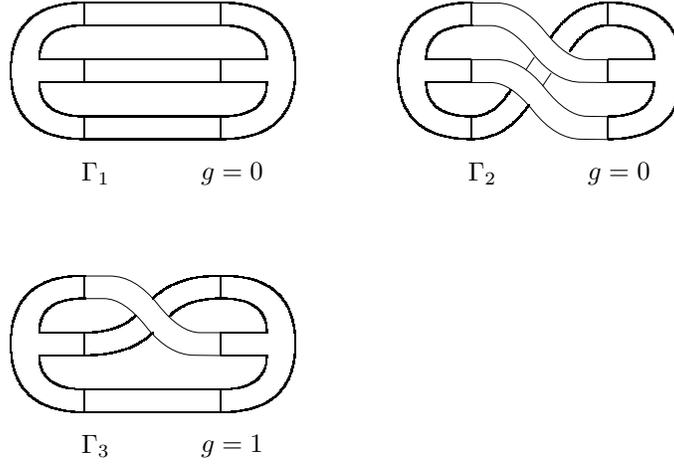
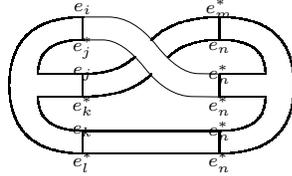


FIGURE 11. Gluing a fat graph from fat flowers

Proof. Let e_i be the standard basis of \mathbb{C}^N , and e_i^* the dual basis. Then the tensor $\text{Tr}(A^m)$ can be written as

$$\text{Tr}(A^m) = \sum_{i_1, \dots, i_m=1}^N (e_{i_1} \otimes e_{i_2}^* \otimes e_{i_2} \otimes e_{i_3}^* \otimes \dots \otimes e_{i_m} \otimes e_{i_1}^*, A^{\otimes m}).$$

One can visualize each monomial in this sum as a labeling of the angle points $P_1, Q_1, \dots, P_m, Q_m$ on the boundary of a fat m -valent flower by $i_1, i_2, i_2, i_3, \dots, i_m, i_1$. Now, the contraction using σ of some set of such monomials is nonzero iff the subscript is constant along each boundary component of $\tilde{\Gamma}_\sigma$ (see Fig. 12). This implies the result. \square



Contraction nonzero iff
 $i = r, j = p, j = m, k = r,$
 $k = p, i = m,$
 that is
 $i = r = k = p = j = m.$

FIGURE 12. Contraction defined by a fat graph.

Let $\tilde{G}_c(\mathbf{n})$ is the set of isomorphism classes of connected fat graphs with n_i i -valent vertices. For $\tilde{\Gamma} \in \tilde{G}_c(\mathbf{n})$, let $b(\tilde{\Gamma})$ be the number of edges minus the number of vertices of the underlying usual graph Γ .

Corollary 4.2.

$$\ln Z_N = \sum_{\mathbf{n}} \left(\prod_i g_i^{n_i} \right) \sum_{\tilde{\Gamma} \in \tilde{G}_c(\mathbf{n})} \frac{N^{\nu(\tilde{\Gamma})} \hbar^{b(\tilde{\Gamma})}}{|\text{Aut}(\tilde{\Gamma})|}.$$

Proof. Let $G_{\text{fat}}(\mathbf{n}) = \prod S_{n_i} \times (\prod \mathbb{Z}/i\mathbb{Z})^{n_i}$. This group acts on T , so that $\tilde{\Gamma}_\sigma = \tilde{\Gamma}_{g\sigma}$, for any $g \in G_{\text{fat}}$ (since cyclic permutations of edges of a flower extend to its fattening). Moreover, the group acts transitively on the set of σ giving a fixed fat graph $\tilde{\Gamma}_\sigma$, and the stabilizer of any σ is $\text{Aut}(\tilde{\Gamma}_\sigma)$. This implies the result. \square

Now for any compact surface Σ with boundary, let $g(\Sigma)$ be the genus of Σ . Then for a connected fat graph $\tilde{\Gamma}$, $b(\tilde{\Gamma}) = 2g(\tilde{\Gamma}) - 2 + \nu(\tilde{\Gamma})$ (minus the Euler characteristic). Thus, defining $\hat{Z}_N(\hbar) = Z_N(\hbar/N)$, we find

Theorem 4.3.

$$\ln \hat{Z}_N = \sum_{\mathbf{n}} \left(\prod g_i^{n_i} \right) \sum_{\tilde{\Gamma} \in \tilde{G}_c(\mathbf{n})} \frac{N^{2-2g(\tilde{\Gamma})} \hbar^{b(\tilde{\Gamma})}}{|\text{Aut}(\tilde{\Gamma})|}.$$

This implies the following important result, due to t'Hooft.

Theorem 4.4. (1) *There exists a limit $W_\infty := \lim_{N \rightarrow \infty} \frac{\ln \hat{Z}_N}{N^2}$. This limit is given by the formula*

$$W_\infty = \sum_{\mathbf{n}} \left(\prod g_i^{n_i} \right) \sum_{\tilde{\Gamma} \in \tilde{G}_c(\mathbf{n})[0]} \frac{\hbar^{b(\tilde{\Gamma})}}{|\text{Aut}(\tilde{\Gamma})|},$$

where $\tilde{G}_c(\mathbf{n})[0]$ denotes the set of **planar** connected fat graphs, i.e. those which have genus zero.

(2) *Moreover, there exists an expansion $\ln \hat{Z}_N / N^2 = \sum_{g \geq 0} a_g N^{-2g}$, where*

$$a_g = \sum_{\mathbf{n}} \left(\prod g_i^{n_i} \right) \sum_{\tilde{\Gamma} \in \tilde{G}_c(\mathbf{n})[g]} \frac{\hbar^{b(\tilde{\Gamma})}}{|\text{Aut}(\tilde{\Gamma})|},$$

and $\tilde{G}_c(\mathbf{n})[g]$ denotes the set of connected fat graphs which have genus g .

Remark 1. Genus zero fat graphs are said to be planar because the underlying usual graphs can be put on the 2-sphere (and hence on the plane) without self-intersections.

Remark 2. t'Hooft's theorem may be interpreted in terms of the usual Feynman diagram expansion. Namely, it implies that for large N , the leading contribution to $\ln(Z_N(\hbar/N))$ comes from the terms in the Feynman diagram expansion corresponding to planar graphs (i.e. those that admit an embedding into the 2-sphere).

4.3. Integration over real symmetric matrices. One may also consider the matrix integral over the space \mathfrak{s}_N of real symmetric matrices of size N . Namely, one puts

$$Z_N = \hbar^{-N(N+1)/4} \int_{\mathfrak{s}_N} e^{-S(A)/\hbar} dA,$$

where S and dA are as above. Let us generalize Theorem 4.4 to this case.

As before, consideration of the large N limit leads to consideration of fat flowers and gluing of them. However, the exact nature of gluing is now somewhat different. Namely, in the Hermitian case we had $(e_i \otimes e_j^*, e_k \otimes e_l^*) = \delta_{il} \delta_{jk}$, which forced us to glue fat flowers preserving orientation. On the other hand, in the real symmetric case $e_i^* = e_i$, and the inner product of the functionals $e_i \otimes e_j$ on the space of symmetric matrices is given by $(e_i \otimes e_j, e_k \otimes e_l) = \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}$. This means that besides the usual (orientation preserving) gluing of fat flowers, we now must allow gluing with a twist of the ribbon by 180° . Fat graphs thus obtained will be called *twisted fat graphs*. That means, a twisted fat graph is a surface with boundary (possibly not orientable), together with a partition into fat flowers, and orientations on each of them (which may or may not match at the cuts, see Fig. 13).

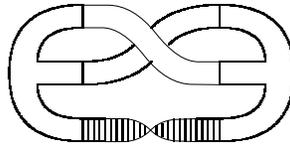


FIGURE 13. Twisted-fat graph

Now one can show analogously to the Hermitian case that the $1/N$ expansion of $\ln \hat{Z}_N$ (where $\hat{Z}_N = Z_N(2\hbar/N)$) is given by the same formula as before, but with summation over the set $\tilde{G}_c^{\text{tw}}(\mathbf{n})$ of twisted fat graphs:

Theorem 4.5.

$$\ln \hat{Z}_N = \sum_{\mathbf{n}} \left(\prod_i g_i^{n_i} \right) \sum_{\tilde{\Gamma} \in \tilde{G}_c^{\text{tw}}(\mathbf{n})} \frac{N^{2-2g(\tilde{\Gamma})} \hbar^{b(\tilde{\Gamma})}}{|\text{Aut}(\tilde{\Gamma})|}.$$

Here the genus g of a (possibly non-orientable) surface is defined by $g = 1 - \chi/2$, where χ is the Euler characteristic. Thus the genus of $\mathbb{R}P^2$ is $1/2$, the genus of the Klein bottle is 1 , and so on.

In particular, we have the analog of t'Hooft's theorem.

Theorem 4.6. (1) *There exists a limit $W_\infty := \lim_{N \rightarrow \infty} \frac{\ln \hat{Z}_N}{N^2}$. This limit is given by the formula*

$$W_\infty = \sum_{\mathbf{n}} \left(\prod_i g_i^{n_i} \right) \sum_{\tilde{\Gamma} \in \tilde{G}_c^{\text{tw}}(\mathbf{n})[0]} \frac{\hbar^{b(\tilde{\Gamma})}}{|\text{Aut}(\tilde{\Gamma})|},$$

where $\tilde{G}_c^{\text{tw}}(\mathbf{n})[0]$ denotes the set of **planar** connected twisted fat graphs, i.e. those which have genus zero.

(2) *Moreover, there exists an expansion $\ln \hat{Z}_N / N^2 = \sum_{g \geq 0} a_g N^{-2g}$, where*

$$a_g = \sum_{\mathbf{n}} \left(\prod_i g_i^{n_i} \right) \sum_{\tilde{\Gamma} \in \tilde{G}_c^{\text{tw}}(\mathbf{n})[g]} \frac{\hbar^{b(\tilde{\Gamma})}}{|\text{Aut}(\tilde{\Gamma})|},$$

and $\tilde{G}_c^{\text{tw}}(\mathbf{n})[g]$ denotes the set of connected twisted fat graphs which have genus g .

Exercise. Consider the matrix integral over the space \mathfrak{q}_N of quaternionic Hermitian matrices. Show that in this case the results are the same as in the real case, except that each twisted fat graph counts with a sign, equal to $(-1)^m$, where m is the number of twistings (i.e. mismatches of orientation at cuts). In other words, $\ln \hat{Z}_N$ for quaternionic matrices is equal $\ln \hat{Z}_{2N}$ for real matrices with N replaced by $-N$.

Hint: use that the unitary group $U(N, \mathbb{H})$ is a real form of $Sp(2N)$, and \mathfrak{q}_N is a real form of the representation of $\Lambda^2 V$, where V is the standard (vector) representation of $Sp(2N)$. Compare to the case of real symmetric matrices, where the relevant representation is $S^2 V$ for $O(N)$, and the case of complex Hermitian matrices, where it is $V \otimes V^*$ for $GL(N)$.

4.4. Application to a counting problem. Matrix integrals are so rich that even the simplest possible example reduces to a nontrivial counting problem. Namely, consider the matrix integral Z_N over complex Hermitian matrices in the case $S(A) = \text{Tr}(A^2)/2 - s \text{Tr}(A^{2m})/2m$, where $s^2 = 0$ (i.e. we work over the ring $\mathbb{C}[s]/(s^2)$). In this case we can set $\hbar = 1$. Then from Theorem 4.4 we get

$$\int_{\mathfrak{h}_N} \text{Tr}(A^{2m}) e^{-\text{Tr}(A^2)/2} dA = P_m(N),$$

where $P_m(N)$ is a polynomial, given by the formula $P_m(N) = \sum_{g \geq 0} \varepsilon_g(m) N^{m+1-2g}$, and $\varepsilon_g(m)$ is the number of ways to glue a surface of genus g from a $2m$ -gon with labeled sides by gluing sides preserving the orientation. Indeed, in this case we have only one fat flower of valency $2m$, which has to be glued with itself; so a direct application of our Feynman rules leads to counting ways to glue a surface of a given genus from a polygon.

The value of this integral is given by the following non-trivial theorem.

Theorem 4.7. (*Harer-Zagier, 1986*)

$$P_m(x) = \frac{(2m)!}{2^m m!} \sum_{p=0}^m \binom{m}{p} 2^p \frac{x(x-1) \dots (x-p)}{(p+1)!}.$$

The theorem is proved in the next subsections.

Looking at the leading coefficient of P_m , we get.

Corollary 4.8. *The number of ways to glue a sphere of a $2m$ -gon is the Catalan number $C_m = \frac{1}{m+1} \binom{2m}{m}$.*

Corollary 4.8 actually has another (elementary combinatorial) proof, which is as follows. For each pairing σ on the set of sides of the $2m$ -gon, let us connect the midpoints of the sides that are paired by straight lines (Fig. 14). It is geometrically evident that if these lines don't intersect then the gluing will give a sphere. We claim that the converse is true as well. Indeed, we can assume that the statement is known for the $2m - 2$ -gon. Let σ be a gluing of the $2m$ -gon that gives a sphere. If there is a connection between two adjacent sides, we may glue them and go from a $2m$ -gon to a $2m - 2$ -gon (Fig. 15). Thus, it is sufficient to consider the case when adjacent sides are never connected. Then there exist adjacent sides a and b whose lines (connecting them to some c, d) intersect with each other. Let us now replace σ by another pairing σ' , whose only difference from σ is that a is connected to b and c to d (Fig. 16). One sees by inspection (check it!) that this does not decrease the number of boundary components of the resulting surface. Therefore, since σ gives a sphere, so does σ' . But σ' has adjacent sides connected, the case considered before, hence the claim.

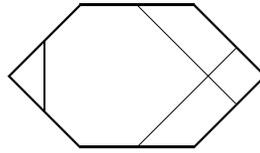


FIGURE 14. Pairing of sides of a 6-gon.

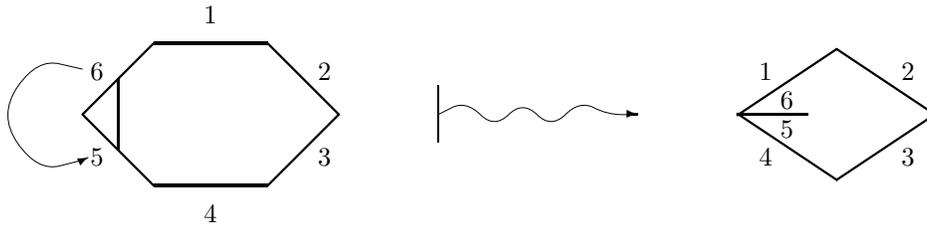


FIGURE 15

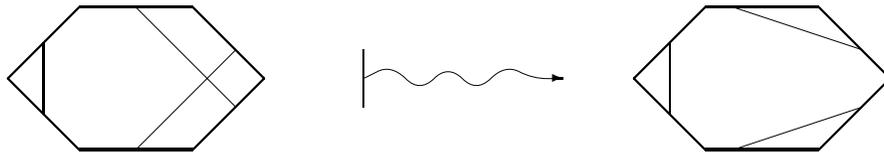


FIGURE 16

Now it remains to count the number of ways to connect midpoints of sides with lines without intersections. Suppose we draw one such line, such that the number of sides on the left of it is $2k$ and on the right is $2l$ (so that $k + l = m - 1$). Then we face the problem of connecting the two sets of $2k$ and $2l$ sides without intersections. This shows that the number of gluings D_m satisfies the recursion

$$D_m = \sum_{k+l=m-1} D_k D_l$$

In other words, the generating function $\sum D_m x^m = 1 + x + \dots$ satisfies the equation $f - 1 = x f^2$. This implies that $f = \frac{1 - \sqrt{1 - 4x}}{2x}$, which yields that $D_m = C_m$. We are done.

Corollary 4.8 can be used to derive the the following fundamental result from the theory of random matrices, discovered by Wigner in 1955.

Theorem 4.9. (*Wigner's semicircle law*) Let f be a continuous function on \mathbb{R} of at most polynomial growth at infinity. Then

$$\lim_{N \rightarrow \infty} \frac{1}{N} \int_{\mathfrak{h}_N} \text{Tr} f(A/\sqrt{N}) e^{-\text{Tr}(A^2)/2} \frac{1}{2\pi} \int_{-2}^2 f(x) \sqrt{4-x^2} dx.$$

This theorem is called the semicircle law because it says that the graph of the density of eigenvalues of a large random Hermitian matrix distributed according to ‘the ‘Gaussian unitary ensemble’ (i.e. with density $e^{-\text{Tr}(A^2)/2} dA$) is a semicircle.

Proof. By Weierstrass uniform approximation theorem, we may assume that f is a polynomial. (Exercise: Justify this step).

Thus, it suffices to check the result if $f(x) = x^{2m}$. In this case, by Corollary 4.8, the left hand side is C_m . On the other hand, an elementary computation yields $\frac{1}{2\pi} \int_{-2}^2 x^{2m} \sqrt{4-x^2} = C_m$, which implies the theorem. \square

4.5. Hermite polynomials. The proof ⁴ of Theorem 4.7 given below uses Hermite polynomials. So let us recall their properties.

Hermite's polynomials are defined by the formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

So the leading term of $H_n(x)$ is $(2x)^n$.

We collect the standard properties of $H_n(x)$ in the following theorem.

Theorem 4.10. (i) The generating function of $H_n(x)$ is $f(x, t) = \sum_{n \geq 0} H_n(x) \frac{t^n}{n!} = e^{2xt-t^2}$.

(ii) $H_n(x)$ satisfy the differential equation $f'' - 2xf' + 2nf = 0$. In other words, $H_n(x)e^{-x^2/2}$ are eigenfunctions of the operator $L = -\frac{1}{2}\partial^2 + \frac{1}{2}x^2$ (Hamiltonian of the quantum harmonic oscillator) with eigenvalues $n + \frac{1}{2}$.

(iii) $H_n(x)$ are orthogonal:

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx = 2^n n! \delta_{mn}$$

(iv) One has

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} x^{2m} H_{2k}(x) dx = \frac{(2m)!}{(m-k)!} 2^{2(k-m)}$$

(if $k > m$, the answer is zero).

(v) One has

$$\frac{H_r^2(x)}{2^r r!} = \sum_{k=0}^r \frac{r!}{2^k k! 2^{r-k} (r-k)!} H_{2k}(x).$$

Proof. (sketch)

(i) Follows immediately from the fact that the operator $\sum (-1)^n \frac{t^n}{n!} \frac{d^n}{dx^n}$ maps a function $g(x)$ to $g(x-t)$.

(ii) Follows from (i) and the fact that the function $f(x, t)$ satisfies the PDE $f_{xx} - 2xf_x + 2tf_t = 0$.

(iii) Follows from (i) by direct integration (one should compute $\int_{\mathbb{R}} f(x, t) f(x, u) e^{-x^2} dx$ using a shift of coordinate).

(iv) By (i), one should calculate $\int_{\mathbb{R}} x^{2m} e^{2xt-t^2} e^{-x^2} dx$. This integral equals

$$\int_{\mathbb{R}} x^{2m} e^{-(x-t)^2} dx = \int_{\mathbb{R}} (y+t)^{2m} e^{-y^2} dy = \sqrt{\pi} \sum_p \binom{2m}{2p} \frac{(2m-2p)!}{2^{m-p}(m-p)!} t^{2p}.$$

The result is now obtained by extracting individual coefficients.

⁴I adopted this proof from D. Jackson's notes

(v) By (iii), it suffices to show that

$$\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} H_r^2(x) H_{2k}(x) e^{-x^2} dx = \frac{2^{r+k} r!^2 (2k)!}{k!^2 (r-k)!}$$

To prove this identity, let us integrate the product of three generating functions. By (i), we have

$$\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} f(x, t) f(x, u) f(x, v) e^{-x^2} dx = e^{2(tu+tv+uv)}.$$

Extracting the coefficient of $t^r u^r v^{2k}$, we get the result. \square

4.6. Proof of Theorem 4.7. We need to compute the integral

$$\int_{\mathfrak{h}_N} \text{Tr}(A^{2m}) e^{-\text{Tr}(A^2)/2} dA.$$

To do this, we note that the integrand is invariant with respect to conjugation by unitary matrices. Therefore, the integral can be reduced to an integral over the eigenvalues $\lambda_1, \dots, \lambda_N$ of A .

More precisely, consider the spectrum map $\sigma : \mathfrak{h}_N \rightarrow \mathbb{R}^N/S_N$. It is well known (due to H. Weyl) that the direct image $\sigma_* dA$ is given by the formula $\sigma_* dA = C e^{-\sum_i \lambda_i^2/2} \prod_{i < j} (\lambda_i - \lambda_j)^2 d\lambda$, where $C > 0$ is a normalization constant that will not be relevant to us. Thus, we have

$$P_m(N) = \frac{\int_{\mathbb{R}^N} (\sum_i \lambda_i^{2m}) e^{-\sum \lambda_i^2/2} \prod_{i < j} (\lambda_i - \lambda_j)^2 d\lambda}{\int_{\mathbb{R}^N} e^{-\sum \lambda_i^2/2} \prod_{i < j} (\lambda_i - \lambda_j)^2 d\lambda}$$

To calculate the integral J_m in the numerator, we will use Hermite's polynomials. Observe that since $H_n(x)$ are polynomials of degree n with highest coefficient 2^n , we have $\prod_{i < j} (\lambda_i - \lambda_j) = 2^{-N(N-1)/2} \det(H_k(\lambda_i))$, where k runs through the set $0, 1, \dots, n-1$. Thus, we find

$$\begin{aligned} J_m &:= \int_{\mathbb{R}^N} (\sum_i \lambda_i^{2m}) e^{-\sum \lambda_i^2/2} \prod_{i < j} (\lambda_i - \lambda_j)^2 d\lambda = \\ &2^{m+N^2/2} N \int_{\mathbb{R}^N} \lambda_1^{2m} e^{-\sum \lambda_i^2} \prod_{i < j} (\lambda_i - \lambda_j)^2 d\lambda = \\ (12) \quad &2^{m-N(N-2)/2} N \int_{\mathbb{R}^N} \lambda_1^{2m} e^{-\sum \lambda_i^2} \det(H_k(\lambda_j))^2 d\lambda = \\ &2^{m-N(N-2)/2} N \int_{\mathbb{R}^N} \lambda_1^{2m} e^{-\sum \lambda_i^2} \left(\sum_{\sigma, \tau \in S_N} (-1)^\sigma (-1)^\tau \prod_i H_{\sigma_i}(\lambda_i) H_{\tau_i}(\lambda_i) \right) d\lambda. \end{aligned}$$

(Here $(-1)^\sigma$ denotes the sign of σ .)

Since Hermite polynomials are orthogonal, the only terms which are nonzero are the terms with $\sigma(i) = \tau(i)$ for $i = 2, \dots, N$. That is, the nonzero terms have $\sigma = \tau$. Thus, we have

$$\begin{aligned} J_m &= 2^{m-N(N-2)/2} N \int_{\mathbb{R}^N} \lambda_1^{2m} e^{-\sum \lambda_i^2} \left(\sum_{\sigma \in S_N} \prod_i H_{\sigma_i}(\lambda_i)^2 \right) d\lambda = \\ (13) \quad &2^{m-N(N-2)/2} N! \gamma_0 \dots \gamma_{N-1} \sum_{j=0}^{N-1} \frac{1}{\gamma_j} \int_{-\infty}^{\infty} x^{2m} H_j(x)^2 e^{-x^2} dx, \end{aligned}$$

where $\gamma_i = \int_{-\infty}^{\infty} H_i(x)^2 e^{-x^2} dx$ are the squared norms of the Hermite polynomials. Applying this for $m = 0$ and dividing J_m by J_0 , we find

$$J_m/J_0 = 2^m \sum_{j=0}^{N-1} \frac{1}{\gamma_j} \int_{-\infty}^{\infty} x^{2m} H_j(x)^2 e^{-x^2} dx$$

Using Theorem 4.10 (iii) and (v), we find: $\gamma_i = 2^i i! \sqrt{\pi}$, and hence

$$J_m/J_0 = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \sum_{j=0}^{N-1} \sum_{k=0}^j \frac{2^m x^{2m} H_{2k}(x)}{2^k k!^2 (j-k)!} e^{-x^2} dx$$

Now, using (iv), we get

$$J_m/J_0 = \frac{(2m)!}{2^m} \sum_{j=0}^{N-1} \sum_{k=0}^j \frac{2^k j!}{(m-k)! k!^2 (j-k)!} =$$

$$\frac{(2m)!}{2^m m!} \sum_{j=0}^{N-1} \sum_{k=0}^j 2^k \binom{m}{k} \binom{j}{k}.$$

The sum over k can be represented as a constant term of a polynomial:

$$\sum_{k=0}^j 2^k \binom{m}{k} \binom{j}{k} = C.T.((1+z)^m (1+2z^{-1})^j).$$

Therefore, summation over j (using the formula for the sum of the geometric progression) yields

$$J_m/J_0 = \frac{(2m)!}{2^m m!} C.T.((1+z)^m \frac{(1+2z^{-1})^N - 1}{2z^{-1}}) = \frac{(2m)!}{2^m m!} \sum_{p=0}^m 2^p \binom{m}{p} \binom{N}{p+1}$$

We are done.