5. The Euler characteristic of the moduli space of curves

Matrix integrals (in particular, computation of the polynomial $P_m(x)$) can be used to calculate the orbifold Euler characteristic of the moduli space of curves. This was done by Harer and Zagier in 1986. Here we will give a review of this result (with some omissions).

5.1. **Euler characteristics of groups.** We start with recalling some basic notions from algebraic topology.

Let Γ be a discrete group, and Y be a contractible finite dimensional CW complex, on which Γ acts cellularly. This means that Γ acts by homeomorphisms of Y that map each cell homeomorphically to another cell. We will assume that the stabilizer of each cell is a finite group (i.e. Y is a proper Γ-complex).

Suppose first that the action of Γ is free (i.e. the stabilizers of cells are trivial). This is equivalent to saying that Γ is torsion free (i.e has no nontrivial finite subgroups), since a finite group cannot act without fixed points on a contractible finite dimensional cell complex (as it has infinite cohomological dimension).

In this case we can define a cell complex Y/Γ (a classifying space for Γ), and we have $H^i(Y/\Gamma, A) = H^i(\Gamma, A)$ for any coefficient group A. In particular, if Y/Γ is finite then Γ has finite cohomological dimension, and the Euler characteristic $\chi(\Gamma) := \sum(-1)^i \dim H^i(\Gamma, \mathbb{Q})$ is equal to $\sum(-1)^i n_i(Y/\Gamma)$, where $n_i(Y/\Gamma)$ denotes the number of cells in Y/Γ of dimension i.

This setting, however, is very restrictive, since it allows only groups of finite cohomological dimension, and in particular excludes all non-trivial finite groups. So let us consider a more general setting: assume that some finite index subgroup $\Gamma' \subset \Gamma$, rather than Γ itself, satisfies the above conditions. In this case, on may define the Euler characteristic of Γ in the sense of Wall, which is the rational number $[\Gamma : \Gamma']^{-1}\chi(\Gamma')$.

It is easy to check that the Euler characteristic in the sense of Wall can be computed using the following *Quillen's formula*

$$\chi(\Gamma) = \sum_{\sigma \in \text{cells(Y)}/\Gamma} \frac{(-1)^{\dim \sigma}}{|\text{Stab}\sigma|}$$

In particular, this number is independent of Γ' (which is also easy to check directly).

Example 1. If G is a finite group then $\chi(G) = |G|^{-1}$ (one takes the trivial group as the subgroup of finite index).

Example 2. $G = SL_2(\mathbb{Z})$. This group contains a subgroup F of index 12, which is free in two generators (check it!). The group F has Euler characteristic -1, since its classifying space Y/F is figure "eight" (i.e., Y is the universal cover of figure "eight"). Thus, the Euler characteristic of $SL_2(\mathbb{Z})$ is $-1/12$.

The Euler characteristic in the sense of Wall has a geometric interpretation in terms of orbifolds. Namely, suppose that Γ is as above (i.e. $\chi(\Gamma)$ is a well defined rational number), and M is a contractible manifold, on which Γ act freely and properly discontinuously. In this case, stabilizers of points are finite, and thus M/Γ is an orbifold. This means, in particular, that to every point $x \in M/\Gamma$ is attached a finite group $\text{Aut}(x)$, of size $\leq [\Gamma : \Gamma']$. Let X_m be the subset of M/Γ, consisting of points x such that $\text{Aut}(x)$ has order m. It often happens that X_m has the homotopy type of a finite cell complex. In this case, the *orbifold Euler characteristic* of M/Γ is defined to be $\chi_{\text{orb}}(M/\Gamma) = \sum_m \chi(X_m)/m$.

Now, we claim that $\chi_{\text{orb}}(M/\Gamma) = \chi(\Gamma)$. Indeed, looking at the projection $M/\Gamma' \to M/\Gamma$, it is easy to see that $\chi_{\text{orb}}(M/\Gamma) = \frac{1}{[\Gamma:\Gamma']}\chi(M/\Gamma')$. But M/Γ' is a classifying space for Γ', so $\chi(M/\Gamma') = \chi(\Gamma')$, which implies the claim.

Example. Consider the group $\Gamma = SL_2(\mathbb{Z})$ acting on the upper half plane H. Then H/Γ is the moduli space of elliptic curves. So as a topological space it is $\mathbb{C}$, where all points have automorphism group $\mathbb{Z}/2$, except the point i having automorphism group $\mathbb{Z}/4$, and ρ which has automorphism group $\mathbb{Z}/6$. Thus, the orbifold Euler characteristic of H/Γ is $(-1)\frac{1}{2} + \frac{1}{4} + \frac{1}{6} = -\frac{1}{12}$. This is not surprising since we proved that $\chi_{\text{orb}}(H/\Gamma) = \chi(\Gamma)$, which was computed to be $-1/12$.

5.2. **The mapping class group.** Now let $g \geq 1$ be an integer, and Σ be a closed oriented surface of genus g. Let $p \in \Sigma$, and let Γ_g^1 be the group of isotopy classes of diffeomorphisms of Σ which preserves

p. We will recall without proof some standard facts about this group, following the paper of Harer and Zagier, Inv. Math., v.85, p. 457-485.

The group Γ_g^1 is not torsion free, but it has a torsion free subgroup of finite index. Namely, consider the homomorphism $\Gamma_g^1 \to Sp(2g, \mathbb{Z}/n\mathbb{Z})$ given by the action of Γ_g^1 on $H_1(\Sigma, \mathbb{Z}/n\mathbb{Z})$. Then for large enough n (in fact, $n \geq 3$), the kernel K_n of this map is torsion free.

It turns out that there exists a contractible finite dimensional cell complex Y, to be constructed below, on which Γ_g^1 acts cellularly with finitely many cell orbits. Thus, the Euler characteristic of Γ_g^1 in the sense of Wall is well defined.

Theorem 5.1. *(Harer-Zagier) $\chi(\Gamma_g^1) = -B_{2g}/2g$, where B_n are the Bernoulli numbers, defined by the generating function $\sum_{n\geq 0} B_n z^n/n! = \frac{z}{e^z-1}$.*

Remark 1. The group Γ_g^1 acts on the Teichmüller space $\mathcal{T}_g^1$, which is, by definition the space of pairs $((R,z), f)$, where (R,z) is a complex Riemann surface with a marked point z, and f is an isotopy class of diffeomorphisms $R \to \Sigma$ that map z to p. One may show that $\mathcal{T}_g^1$ is a contractible manifold of dimension $6g-4$, and that the action of Γ_g^1 on $\mathcal{T}_g^1$ is properly discontinuous. In particular, we may define an orbifold $M_g^1 = \mathcal{T}_g^1/\Gamma_g^1$. This orbifold parameterizes pairs (R,z) as above; therefore, it is called the moduli space of Riemann surfaces (=smooth complex projective algebraic curves) with one marked point. Thus, Theorem 5.1 gives the orbifold Euler characteristic of the moduli space of curves with one marked point.

Remark 2. If $g > 1$, one may define the analogs of the above objects without marked points, namely the mapping class group Γ_g, the Teichmüller space $\mathcal{T}_g$, and the moduli space of curves $M_g = \mathcal{T}_g/\Gamma_g$ (one can do it for $g = 1$ as well, but in this case there is no difference with the case of one marked point, since the translation group allows one to identify any two points on Σ). It is easy to see that we have an exact sequence $1 \to \pi_1(\Sigma) \to \Gamma_g^1 \to \Gamma_g \to 1$, which implies that $\chi(\Gamma_g) = \chi(\Gamma_g^1)/\chi(\Sigma)$. Thus, $\chi(\Gamma_g) = \chi_{orb}(M_g) = B_{2g}/4g(g-1)$

5.3. **Construction of the complex Y.** We begin the proof of Theorem 5.1 with the construction of the complex Y, following the paper of Harer and Zagier. We will first construct a simplicial complex with a Γ action, and then use it to construct Y.

Let $(\alpha_1, \ldots, \alpha_n)$ be a collection of closed simple unoriented curves on Σ, which begin and end at p, and do not intersect other than at p. Such a collection is called an arc system if two conditions are satisfied:

(A) none of the curves is contractible to a point;

(B) none of the curves is contractible to each other.

Define a simplicial complex A, whose $n-1$-simplices are isotopy classes of arc systems consisting of $n \geq 1$ arcs, and the boundary of a simplex corresponding to $(\alpha_1, \ldots \alpha_n)$ is the union of simplices corresponding to the arc system $(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_n)$ (α_i is omitted).

It is clear that the group Γ_g^1 acts simplicially on A.

Example. Let $g = 1$, i.e. $\Sigma = S^1 \times S^1$. Then $\Gamma_g^1 = SL_2(\mathbb{Z})$. Up to its action, there are only three arc systems (Fig. 17). Namely, viewing S^1 as the unit circle in the complex plane, and representing arcs parametrically, we may write these three systems as follows:

$$B_0 = \{(e^{i\theta}, 1)\}; B_1 = \{(e^{i\theta}, 1), (1, e^{i\theta})\}; B_2 = \{(e^{i\theta}, 1), (1, e^{i\theta}), (e^{i\theta}, e^{i\theta})\}$$

¿From this it is easy to find the simplicial complex A. Namely, let T be the tree with root t_0 connected to three vertices t_1, t_2, t_3, with each t_i connected to two vertices t_{i1}, t_{i2}, each t_{ij} connected to t_{ij1}, t_{ij2}, etc. (Fig. 18). Put at every vertex of T a triangle, with sides transversal to the three edges going out of this vertex, and glue triangles along the sides. This yields the complex A, Fig. 19 (check it!). The action of $SL_2(\mathbb{Z})$ (or rather $PSL_2(\mathbb{Z})$) on this complex is easy to describe. Namely, recall that $PSL_2(\mathbb{Z})$ is generated by S, U such that $S^2 = U^3 = 1$. The action of S, U on T is defined as follows: S is reflection with flip with respect to a side of the triangle Δ_0 centered at t_0 (Fig. 20), and U is rotation by $2\pi/3$ around t_0.

This example shows that the action of Γ_g^1 on A is not properly discontinuous, and some simplices have infinite stabilizers (in the example, it is the 0-dimensional simplices). Thus, we would like to throw away the "bad" simplices. To do it, let us say that an arc system $(\alpha_1, .., \alpha_n)$ fills up Σ if it cuts

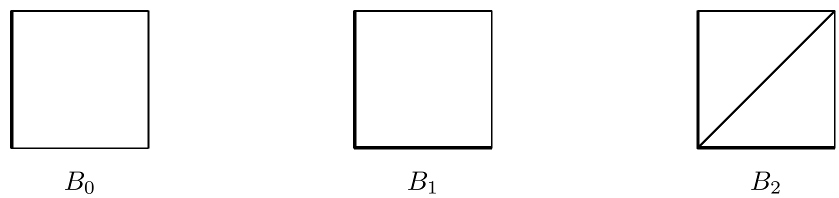

FIGURE 17. Three arc systems.

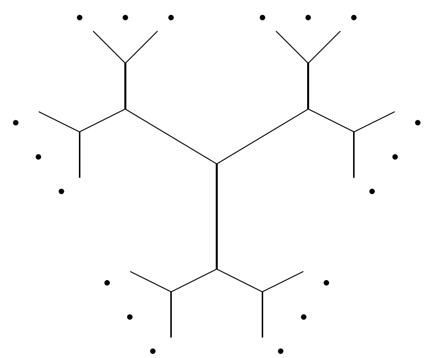

FIGURE 18. Tree T

Σ into a union of regions diffeomorphic to the open disk. Let A_∞ be the union of the simplices in A corresponding to arc systems that do not fill up Σ. This is a closed subset, since the property of not filling up Σ is obviously stable under taking an arc subsystem. Thus, $A \setminus A_\infty$ is an open subset of A. In the example above, it is the complex A with 0-dimensional simplices removed.

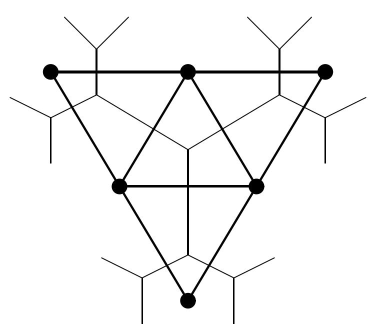

FIGURE 19. Complex A

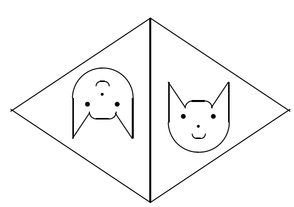

FIGURE 20. Reflection with a flip.

The following theorem shows that $A \setminus A_\infty$ is in fact a combinatorial model for the Teichmüller space $\mathcal{T}_g^1$, with the action of Γ_g^1.

Theorem 5.2. *(Mumford) (a) The action of Γ_g^1 on $A \setminus A_\infty$ is properly discontinuous.*

(b) $A \setminus A_\infty$ is a topologically a manifold, which is Γ_g^1-equivariantly homeomorphic to the Teichmüller space $\mathcal{T}_g^1$; in particular, it is contractible.

Remark 1. Theorem 5.2 exhibits the significance of conditions (A) and (B). Indeed, if either of the conditions were dropped, then one could consider arc systems $(\alpha_1, \ldots, \alpha_n)$ with arbitrarily large n, while with conditions (A),(B), as seen from Theorem 5.2, the largest value of n is $6g-3$.

Remark 2. If $g = 1$, Theorem 5.2 is clear from the explicit description of A (convince yourself of this!).

Theorem 5.2 is rather deep, and we will not give its proof, which is beyond the scope of this course. Rather, we will use it to define the "Poincare dual" CW complex Y of $A \setminus A_\infty$. Namely, to each filling arc system $(\alpha_1, \ldots, \alpha_n)$ we will assign a $6g-3-n$-dimensional cell, and the boundary relation is opposite to the one before. The existence of this CW complex follows from the fact that $A \setminus A_\infty$ is a manifold. For instance, in the case $g = 1$ the complex Y is the tree T.

Now, the complex Y is contractible (since so is $A \setminus A_\infty$), and admits a cellular action of Γ_g^1 with finitely many cell orbits and finite stabilizers. This means that the Euler characteristic of Γ_g^1 is given by Quillen's formula.

$$\chi(\Gamma_g^1) = \sum_{\sigma \in \mathrm{cells}(Y)/\Gamma_g^1} (-1)^{\dim \sigma} \frac{1}{|\mathrm{Stab}\sigma|}.$$

Example. In the $g = 1$ case, T has one orbit of 0-cells and one orbit of 1-cells. The stabilizer of a 0-cell in $SL_2(\mathbb{Z})$ is $\mathbb{Z}/6$, and of 1-cell is $\mathbb{Z}/4$. Hence, $\chi(SL_2(\mathbb{Z})) = \frac{1}{6} - \frac{1}{4} = -\frac{1}{12}$, which was already computed before by other methods.

5.4. **Enumeration of cells in Y/Γ_g^1.** Now it remains to count cells in Y/Γ_g^1, i.e. to enumerate arc systems which fill Σ (taking into account signs and stabilizers) To do this, we note that by definition of "filling", any filling arc system S defines a cellular decomposition of Σ. Thus, let S^* be the Poincare dual of this cellular decomposition. Since S has a unique zero cell, S^* has a unique 2-cell. Let n be the number of 1-cells in S (or S^*). Then (Σ, S^*) is obtained by gluing a $2n$-gon (=the unique 2-cell) according to a pairing of its sides preserving orientation. (Note that S can be reconstructed as $(S^*)^*$).

This allows us to link the problem of enumerating filling arc systems with the problem of counting such gluings, which was solved using matrix integrals. Namely, the problem of enumerating filling arc systems is essentially solved modulo one complication: because of conditions (A) and (B) on an arc system, the gluings we will get will be not arbitrary gluings, but gluings which also must satisfy some conditions. Namely, we have

Lemma 5.3. *Let $(\alpha_1, \ldots, \alpha_n)$ be a system of curves, satisfying the axioms of a filling arc system, except maybe conditions (A) and (B). Then*

(i) $(\alpha_1, \ldots, \alpha_n)$ satisfies condition (A) iff no edge in the corresponding gluing is glued to a neighboring edge.

(ii) $(\alpha_1, \ldots, \alpha_n)$ satisfies condition (B) iff no two consecutive edges are glued to another pair of consecutive edges in the opposite order.

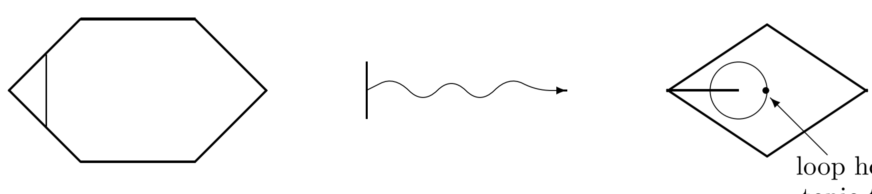

FIGURE 21

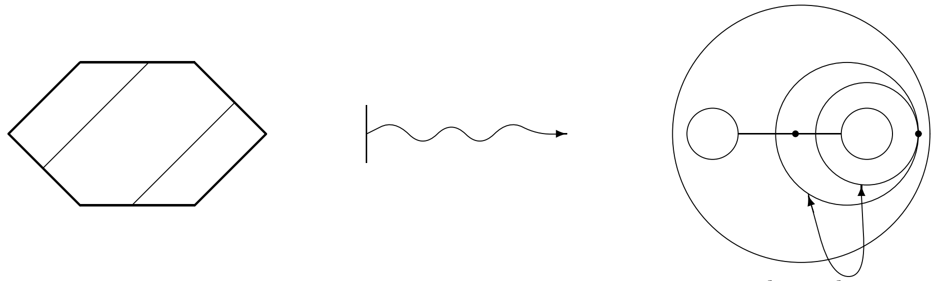

FIGURE 22

The lemma is geometrically evident, and its proof is obtained by drawing a picture (Fig. 21 for (i), Fig. 22 for (ii)). Motivated by the lemma, we will refer to the conditions on a gluing in (i) and (ii) also as conditions (A) and (B).

Denote by $\varepsilon_g(n), \mu_g(n), \lambda_g(n)$ the numbers of gluings of a (labeled) $2n$-gon into a surface of genus g, with no conditions, condition (A), and conditions (A),(B), respectively (so $\varepsilon_g(n)$ is the quantity we already studied).

Proposition 5.4. *One has*

$$\chi(\Gamma_g^1) = \sum_n (-1)^{n-1} \frac{\lambda_g(n)}{2n}.$$

Proof. Each filling arc system σ arises from $2n/|\mathrm{Stab}(\sigma)|$ gluings (since the labeling of the polygon does not matter for the resulting surface with an arc system). Thus, the result follows from Quillen's formula. □

5.5. **Computation of $\sum_n (-1)^{n-1} \frac{\lambda_g(n)}{2n}$.** Now it remains to compute the sum on the right hand side. To do this, we will need to link $\lambda_g(n)$ with $\varepsilon_g(n)$, which has already been computed. This is accomplished by the following lemma.

Lemma 5.5. *One has*

$$\varepsilon_g(n) = \sum_i \binom{2n}{i} \mu_g(n-i);$$

$$\mu_g(n) = \sum_i \binom{n}{i} \lambda_g(n-i).$$

Proof. Proof of the first equation. Let σ be a gluing of a $2n$-gon Δ with labeled vertices. If there is a pair of consecutive edges that are glued, we can glue them to obtain a $2n-2$-gon. Proceeding like this as long as we can, we will arrive at a $2n-2i$-gon Δ_σ, with a gluing σ' of its sides which satisfies condition (A). Note that Δ_σ and σ' do not depend on the order in which neighboring edges were glued to each other, and Δ_σ has a canonical labeling by $1, \ldots, 2n-2i$, in the increasing order of the "old" labels. Now, we claim that each (Δ_σ, σ') is obtained in exactly $\binom{2n}{i}$ ways; this implies the required statement.

Indeed, let us consider vertices of Δ that ended up in the interior of Δ_σ. They have mapped to i points in the interior (each gluing of a pair of edges produces a new point). Let us call these points $w_1, \ldots, w_i$, and let ν_j be the smallest label of a vertex of Δ that goes to w_j. Then $\nu_1, \ldots, \nu_i$ is a subset of $\{1, \ldots, 2n\}$. This subset completely determines the gluing σ if (Δ_σ, σ') are given: namely, we should choose ν_j such that $\nu_j + 1 \neq \nu_k$ for any k, and glue the two edges adjacent to ν_j; then relabel by $1, \ldots, 2n-2$ the remaining vertices (in increasing order of "old" labels), and continue the step again, and so on. From this it is also seen that any set of ν_j may arise. This proves the claim.

Proof of the second equation. Let σ be a gluing of Δ (with labeled edges) which satisfies condition (A) but not necessarily (B). If a_1, a_2 are consecutive edges that are glued to consecutive edges b_2, b_1

in the opposite order, then we may unite a_1, a_2 into a single edge a, and b_2, b_1 into b, and obtain a $2n-2$-gon with a gluing. Continuing so as long as we can, we will arrive at a $2n-2i$-gon Δ_σ with a new gluing σ', which satisfies conditions (A) and (B). In Δ_σ, each (j-th) pair of edges is obtained for m_j+1 pairs of edges in Δ. Thus, $\sum_{j=1}^{m-i} m_j = i$. Furthermore, for any (Δ_σ, σ') the collection of numbers $m_1, \ldots, m_{n-i}$ defines (Δ, σ) uniquely, up to deciding which of the m_1+1 edges constituting the first edge of Δ_σ should be labeled by 1. Thus, each (Δ_σ, σ') arises in the number of ways given by the formula

$$\sum_{m_1,\ldots,m_{n-i}:\sum m_j = i} (m_1+1).$$

It is easy to show (check!) that this number is equal to $\binom{n}{i}$. The second equation is proved. □

The completion of the proof of Theorem 5.1 depends now on the following computational lemma.

Lemma 5.6. *Let $\varepsilon(n), \mu(n), \lambda(n)$, $n \geq 0$, be sequences satisfying the equations*

$$\varepsilon(n) = \sum_i \binom{2n}{i} \mu(n-i);$$

$$\mu(n) = \sum_i \binom{n}{i} \lambda(n-i).$$

Assume also that $\varepsilon(n) = \binom{2n}{n} f(n)$, where f is a polynomial such that $f(0) = 0$. Then $\lambda(0) = 0$, $\lambda(n)$ has finitely many nonzero values, and $\sum (-1)^{n-1}\lambda(n)/2n = f'(0)$.

Proof. Let us first consider any sequences $\varepsilon(n)$, $\mu(n)$, and $\lambda(n)$ linked by the equations of the lemma. Let $E(z)$, $M(z)$, and $L(z)$ be their generating functions (i.e. $E(z) = \sum_{n\geq 0} \varepsilon(n) z^n$ etc.). We claim that

$$E(z) = \frac{1+\sqrt{1-4z}}{2(1-4z)} L\left(\frac{1-\sqrt{1-4z}}{2\sqrt{1-4z}}\right).$$

To see this, it suffices to consider the case $\lambda_i = \delta_{ki}$ for some k. In this case,

$$E(z) = \sum_{i,n} \binom{2n}{i}\binom{n-i}{k} z^n = \sum_{p,q\geq 0} \binom{2p+2q}{p}\binom{q}{k} z^{p+q}$$

But the function

$$F_r(z) := \sum_{p\geq 0} \binom{2p+r}{p} z^p$$

equals

$$F_r(z) = \frac{1}{\sqrt{1-4z}} \left(\frac{1-\sqrt{1-4z}}{2z}\right)^r,$$

as may be easily seen by induction from the recursion $F_r = z^{-1}(F_{r-1} - F_{r-2})$, $r \geq 2$. Substituting this in the formula for $E(z)$, one gets (after trivial simplifications)

$$E(z) = \frac{1+\sqrt{1-4z}}{2(1-4z)} \left(\frac{1-\sqrt{1-4z}}{2\sqrt{1-4z}}\right)^k,$$

as desired.

Now assume that $\varepsilon(n)$ satisfies the polynomiality condition. This means that $E(z) = P(z\partial)|_{z=0}\frac{1}{\sqrt{1-4z}}$, where P is a polynomial with vanishing constant term. To prove our claim, it suffices to consider the case $P(z) = (1+a)^z - 1$, where a is a formal parameter (so $P'(0) = \ln(1+a)$). In this case we get

$$E(z) = \frac{1}{\sqrt{1-4(1+a)z}} - \frac{1}{\sqrt{1-4z}}$$

Hence,

$$L(u) = \frac{1}{1+u}\left(\frac{1}{\sqrt{1-4au(1+u)}} - 1\right)$$

Therefore,

$$\sum_k (-1)^{k-1}\lambda_k/2k = \frac{1}{2}\int_{-1}^{0} L(u)du/u = \frac{1}{2}\sum_{p\geq 1}\binom{2p}{p}(-1)^{p-1}a^p\int_0^1 x^{p-1}(1-x)^{p-1}dx.$$

But $\int_0^1 x^{p-1}(1-x)^{p-1}dx$ is a beta integral, and it equals $(p-1)!^2/(2p-1)!$. Thus, the above integral equals $\sum_{p\geq 1}(-1)^{p-1}a^p/p = \ln(1+a)$, as desired. □

Now we finish the proof of the theorem. Recall that using matrix integrals we have proved the formula

$$P_n(x) := \sum_g \varepsilon_g(n)x^{n+1-2g} = \frac{(2n)!}{2^n n!}\sum_{p\geq 0}\binom{n}{p}2^p\binom{x}{p+1} \tag{14}$$

Let us set $q = n-p$. Then expression (14) takes the form

$$P_n(x) = \binom{2n}{n}\sum_{q\geq 0}2^{-q}\binom{n}{q}\frac{n!}{(n-q+1)!}x(x-1)\cdots(x-n+q). \tag{15}$$

We claim now that the coefficient of x^{-2g} $(g\geq 1)$ in the polynomial $P_n(x)/x^{n+1}$ are of the form $\binom{2n}{n}f_g(n)$, where f_g is a polynomial. Indeed, contributions to the coefficient of x^{-2g} come from terms with $q\leq 2g$ only, so it suffices to check that each of these contributions is as stated. This reduces to checking that the coefficients of the Laurent polynomial $Q(x,n) = (1-1/x)\cdots(1-n/x)$ are polynomials in n, which vanish at -1 (except, of course, the leading coefficient). To see this, let $Q(x,a) = \frac{\Gamma(x)}{\Gamma(x-a)x^a}$ (this equals to $Q(x,n)$ if $a=n$). This function has an asymptotic Taylor expansion in $1/x$ as $x\to+\infty$, and it is easy to show that the coefficients are polynomials in a. Moreover, $Q(x,-1)=1$, which implies the required statement.

Furthermore, we claim that $f_g(0)=0$: again, this follows from the fact that the non-leading coefficients of the expansion of $Q(x,a)$ vanish at $a=0$. But this is clear, since $Q(x,0)=1$.

Thus, we are in a situation where Lemma 5.6 can be applied. So it remains to compute $\sum_{g\geq 1}f_g'(0)x^{-2g}$. To do this, observe that the terms with $q>1$ do not contribute to $f_g'(0)$, as they are given by polynomials of n that are divisible by n^2. So we only need to consider $q=0$ and $q=1$. For $q=1$, the contribution is the value of $(2x)^{-1}(1-1/x)\dots(1-n/x)$ at $n=0$, i.e. it is $1/2x$. For $q=0$, the contribution is the derivative at 0 with respect to n of $(1-1/x)\cdots(1-n/x)/(n+1)$, i.e. it is $\frac{d}{da}|_{a=0}\frac{Q(x,a)}{a+1} = -1+\frac{d}{da}|_{a=0}Q(x,a)$. Thus, we have (asymptotically)

$$\sum_{g\geq 1}f_g'(0)x^{-2g} = \frac{1}{2x}+\frac{d}{da}|_{a=0}Q(x,a) = \frac{1}{2x}+\frac{\Gamma'(x)}{\Gamma(x)}-\log x$$

However, the classical asymptotic expansion for Γ'/Γ is:

$$\frac{\Gamma'(x)}{\Gamma(x)} = \log x - \frac{1}{2x} - \sum_{g=1}^{\infty}\frac{B_{2g}}{2g}x^{-2g}.$$

This implies that $f_g'(0) = -B_{2g}/2g$. Hence the Harer-Zagier theorem is proved.