6. Matrix integrals and counting planar diagrams

6.1. **The number of planar gluings.** Let us return to the setting of §4. Thus, we have a potential $U(x) = x^2/2 - \sum_{j\geq 0} g_j x^j/j$ (with g_j being formal parameters), and consider the matrix integral

$$Z_N(\hbar) = \hbar^{-N/2} \int_{\mathfrak{h}_N} e^{-TrU(A)} dA.$$

Let $\hat{Z}_N(\hbar) = Z_N(\hbar/N)$. We have seen that

$$\lim_{N\to\infty} \frac{\ln \hat{Z}_N}{N^2} = W_\infty,$$

where W_∞ is given by summation over planar fat graphs:

$$W_\infty = \sum_{\mathbf{n}} \prod_i g_i^{n_i} \sum_{\widetilde{\Gamma}\in \widetilde{G}_c(\mathbf{n})[0]} \frac{\hbar^{b(\widetilde{\Gamma})}}{|\mathrm{Aut}(\widetilde{\Gamma})|}$$

In particular, the coefficient of $\prod g_i^{n_i}$ is (up to a power of $\hbar$) the number of (orientation preserving) gluings of a fat graph of genus zero out of a collection of fat flowers containing n_i i-valent flowers for each i, divided by $\prod i^{n_i} n_i!$.

On the other hand, one can compute W_∞ explicitly as a function of g_i by reducing the matrix integral to an integral over eigenvalues, and then using a fundamental fact from the theory of random matrices: the existence of an asymptotic distribution of eigenvalues as $N \to \infty$. This approach allows one to obtain simple closed formulas for the numbers of planar gluings, which are quite nontrivial and for which direct combinatorial proofs were discovered only very recently.

To illustrate this method, we will restrict ourselves to the case of the potential $U(x) = x^2/2 + gx^4$ (so $g_4 = -4g$ and other $g_i = 0$), and set $\hbar = 1$. Then $W_\infty = \sum_{n\geq 1} c_n(-g)^n/n!$, where c_n is a number of connected planar gluings of a set of n 4-valent flowers. In other words, c_n is the number of ways (up to isotopy) to connect n "crosses" in the 2-sphere so that all crosses are connected with each other, and the connecting lines do not intersect.

Exercise. Check by drawing pictures that $c_1 = 2$, $c_2 = 36$.

Theorem 6.1. *(Brezin, Itzykson, Parisi, Zuber, 1978). One has*

$$c_n = (12)^n(2n-1)!/(n+2)!$$

6.2. **Proof of Theorem 6.1.** Let us present the proof of this theorem (with some omissions). We will assume that g is a positive real number, and compute the function $W_\infty(g)$ explicitly. We follow the paper of Brezin, Itzykson, Parisi, and Zuber "Planar diagrams", Comm. Math. Phys. 59, p. 35-51, 1978.

The relevant matrix integral has the form

$$\hat{Z}_N = \int_{\mathfrak{h}_N} e^{-N\mathrm{Tr}(A^2/2+gA^4)} dA.$$

Passing to eigenvalues, we get

$$\hat{Z}_N = \frac{J_N(g)}{J_N(0)},$$

where

$$J_N(g) = \int_{\mathbb{R}^N} e^{-N(\sum \lambda_i^2/2 + g\sum \lambda_i^4)} \prod_{i<j} (\lambda_i - \lambda_j)^2 d\lambda.$$

Thus, $W_\infty(g) = E(g) - E(0)$, where $E(g) = \lim_{N\to\infty} N^{-2} \ln J_N(g)$.

Proposition 6.2. *(steepest descent principle) $E(g)$ equals the maximal value of the logarithm of the integrand.*

The proposition says, essentially, that the integrand has a sufficiently sharp maximum, so that the leading behavior of the integral can be computed by the steepest descent formula. We note that we cannot apply the steepest descent formula without explanations, since the integral is over a space whose dimension grows as the perturbation parameter $1/N$ goes to 0. In other words, it is necessary to do some estimates which we will omit. We will just mention that for $g = 0$, this result can be derived from the explicit evaluation of the integral using Hermite polynomials (see §4). For the general case, we refer the reader to the book of P. Deift "Orthogonal polynomials and random matrices: a Riemann-Hilbert approach".

The integrand $K(\lambda_1, \dots, \lambda_N) = e^{-N(\sum \lambda_i^2/2 + g \sum \lambda_i^4)} \prod_{i<j} (\lambda_i - \lambda_j)^2$ has a unique maximum, because it is logarithmically concave (check it!). The maximum of the integrand is found by equating the partial derivatives to zero. This yields

$$\sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} = N(\frac{1}{2}\lambda_i + 2g\lambda_i^3). \tag{16}$$

Let $\lambda_1 < \lambda_2 < \cdots < \lambda_N$ be the unique (up to permutations) solution of this system of equations.

Proposition 6.3. *The normalized counting measures* $\frac{1}{N}\sum \delta(x - \lambda_i)$ *converge weakly to a measure* $\mu(x) = f(x, g)dx$*, where* $f(x, g)$ *is a continuous function, supported on a finite interval* $[-2a, 2a]$*, and differentiable on this interval.*

For the proof we again refer the reader to the book of P. Deift (p. 132 and later). We note that for $g = 0$, by Wigner's semicircular law, $a = 1$ and $f(x, 0) = \frac{1}{2\pi}\sqrt{4 - x^2}$; so $f(x, g) = \frac{1}{2\pi}\sqrt{4 - x^2} + O(g)$.

Now our job will be to find the function $f(x, g)$. Passing to the limit in equation 16 (which requires justification that we will omit), we get

$$\int_{-2a}^{2a} \frac{1}{y - x} f(x, g)dx = \frac{1}{2}y + 2gy^3, \ |y| \leq 2a$$

where the integral is understood in the sense of principal value.

This is a linear integral equation on $f(x, g)$, which can be solved in a standard way. Namely, one considers the analytic function $F(y) = \int_{-2a}^{2a} \frac{1}{y-x} f(x, g)dx$ for y in the complex plane but outside of the interval $[-2a, 2a]$. For $y \in [-2a, 2a]$, let $F_+(y)$, $F_-(y)$ denote the limits of $F(y)$ from above and below. Then by the Plemelj formula, the integral equation implies

$$\frac{1}{2}(F_+(y) + F_-(y)) = \frac{1}{2}y + 2gy^3.$$

On the other hand, $F_+(y) = \overline{F_-(y)}$. Hence, $\mathrm{Re}F_+(y) = \mathrm{Re}(F_-(y)) = \frac{1}{2}y + 2gy^3$.

Now set $y = a(z + z^{-1})$. Then, as y runs through the exterior of $[-2a, 2a]$, z runs through the exterior of the unit circle. So the function $G(z) := F(y)$ is analytic on the outside of the unit circle, with decay at infinity, and $\mathrm{Re}G(z) = \frac{1}{2}a(z + z^{-1}) + 2ga^3(z + z^{-1})^3$, $|z| = 1$. This implies that $G(z)$ is twice the sum of all negative degree terms of this Laurent polynomial. In other words, we have

$$G(z) = 4ga^3z^{-3} + (a + 12ga^3)z^{-1}.$$

This yields

$$F(y) = \frac{1}{2}y + 2gy^3 - (\frac{1}{2} + 4ga^2 + 2gy^2)\sqrt{y^2 - 4a^2}.$$

Now $f(y, g)$ is found as the jump of F:

$$f(y, g) = \frac{1}{\pi}(\frac{1}{2} + 4ga^2 + 2gy^2)\sqrt{4a^2 - y^2}.$$

It remains to find a in terms of g. We have $yF(y) \to 1$, $y \to \infty$ (as $\int f(x, g)dx = 1$), hence $zG(z) \to 1/a$, $z \to \infty$. This yields $1/a = a + 12ga^3$, or

$$12ga^4 + a^2 - 1 = 0.$$

This allows one to determine a uniquely:

$$a = \Big(\frac{(1 + 48g)^{1/2} - 1}{24g}\Big)^{1/2}.$$

Now let us calculate $E(g)$. It follows from the above that

$$E(g) = \int_{-2a}^{2a} \int_{-2a}^{2a} \ln|x-y| f(x,g) f(y,g) dx dy - \int_{-2a}^{2a} (\frac{1}{2}x^2 + gx^4) f(x,g) dx.$$

On the other hand, let us integrate the integral equation defining $f(x,g)$ with respect to y (from 0 to u). Then we get

$$2 \int_{-2a}^{2a} (\ln|x-u| - \ln|x|) f(x,g) dx = \frac{1}{2}u^2 + gu^4.$$

Substituting this into the expression for $E(g)$, we get

$$E(g) = \int_{-2a}^{2a} (\ln|u| - \frac{1}{4}u^2 - \frac{1}{2}gu^4) f(u,g) du$$

Since $f(u,g)$ is known, this integral can be computed. In fact, can be expressed via elementary functions, and after calculations we get

$$E(g) - E(0) = \ln a - \frac{1}{24}(a^2-1)(9-a^2).$$

Substituting here the expression for a, after a calculation one finally gets:

$$E(g) - E(0) = \sum_{k=1}^{\infty} (-12g)^k \frac{(2k-1)!}{k!(k-2)!}.$$

This implies the required formula for c_n.