

Lecture 10

(Finished asymptotics of Bessel functions from previous lecture notes.)

Discussed **boundary conditions** more generally than we have done in the past. Up to now, we have mostly considered $u=0$ (Dirichlet) or $\mathbf{n}\cdot\nabla u=0$ (Neumann) on the boundary, and mostly the former.

More generally, we can consider "general Dirichlet" and "general Neumann" boundary conditions, where either the values $u(x)$ or the normal derivatives $\mathbf{n}\cdot\nabla u$ are some given function $g(\mathbf{x},t)$ on the boundary. (For example, general Dirichlet boundary conditions arise for a drum head where the edges are not held flat, and you may even be warping the edges as a function of time.) If $g\neq 0$, these functions u do *not* form a vector space because they do not include $u=0$, so we must transform the problem somehow to get back to linear algebra.

For general Dirichlet, one simple approach is to write $u=v+g$, where our new unknown v has zero-Dirichlet boundaries (similar to pset 1). Showed how this transforms e.g. a wave equation $\hat{A}u=\partial^2 u/\partial t^2-f(x,t)$ into wave equation in v but with an additional "force" term modifying f : $\hat{A}v=\partial^2 v/\partial t^2-[f+\hat{A}g-\partial^2 g/\partial t^2]$. For example, considered the steady-state 1d version of this problem $d^2u/dx^2=0$ (Laplace's equation) with $u(0)=a$, $u(L)=b$ boundaries and showed that the solution is a straight line, which is physically obvious for a string stretched from $(0,a)$ to (L,b) .

Intuitively, this makes a certain amount of sense: warping the boundary corresponds to an external force. But intuitively, the "physical" boundary force is only applied at the boundary, not everywhere in Ω as it is for a general $g(x)$ above. It turns out that we can do this, too. It is easier to see this in the discrete case, for the same 1d problem as above. In this case, showed that we obtained the same Dirichlet A matrix as we do for 0 boundary conditions, while the (a,b) boundary conditions just turned into terms added to the right hand side, but only in the first and last rows: an "external force" applied at the boundaries. The PDE version of this technique involves delta functions, which we aren't prepared to handle yet. (In fact, this generalizes to cases where we want to specify jumps and other discontinuities of u in the *interior* of Ω as well, in which case one can again use new surface-localized terms on the right-hand-side and it is sometimes called an "immersed boundary" or "imbedded boundary" method, especially in fluid mechanics.)

To better understand how Neumann boundary conditions arise, we have to better understand the meaning of ∇u . Considered the case of a diffusion equation, where u =mass/volume of some solute, and $\nabla\cdot c\nabla u=\partial u/\partial t$. The total mass M within some volume V is just $\int_V u$, and showed by applying the divergence theorem we obtain dM/dt equal to a surface integral of $c\nabla u$. Since $dM/dt>0$ when mass is flowing *in* to the volume, this means that $-c\nabla u$ is a mass "flux" vector (mass/time·area).

If we have diffusion in a closed container, so that no mass can flow in or out of Ω , we then immediately see that we should apply (0) Neumann boundary conditions. Furthermore, total mass $=\int_{\Omega}u$ is *conserved* (constant in time) for any solution u .

More generally, for any equation $\hat{A}u = \partial u / \partial t$, showed that we obtain a **conservation law** $\partial / \partial t \langle v, u \rangle = 0$ for any $v(\mathbf{x})$ in the *left null space* $N(\hat{A}^*)$.

For the case of diffusion with Neumann boundary conditions, reviewed the fact that $\hat{A} = \hat{A}^*$ but \hat{A} is only negative *semidefinite*: $N(\hat{A}) = N(\hat{A}^*)$ contains any *constant function*, and is spanned by $v(\mathbf{x}) = 1$. Hence $\langle 1, u \rangle$ is conserved. i.e. total mass, or total heat, or average temperature, is conserved in a closed/insulated Ω .

Another example of a (0) Neumann boundary condition arises when we are considering $u(\mathbf{x})$ that are mirror-symmetric (even) around some mirror plane, which is equivalent to imposing a Neumann boundary condition on the mirror plane. (Similar, antisymmetric/odd symmetry is equivalent to a zero Dirichlet boundary.) Another example is a stretched string where one end can slide freely up and down a rod with no friction: that end has a Neumann condition.

There are many other possible boundary conditions, of course. The most complicated ones can arise for PDEs with multiple unknowns (e.g. pressure, temperature, velocity, ...), in which case the boundary conditions may be equations relating several different unknowns or their derivatives.

One can also have *nonlocal* boundary conditions, in which u at one point on $\partial\Omega$ is related to u at a *different* point. The most common example of this are *periodic* boundary conditions. e.g. considered $\hat{A} = d^2/dx^2$ on $[0, L]$ for $u(0) = u(L)$. Showed that \hat{A} is still self-adjoint, but not because the boundary terms are *individually* zero, but rather because the $x=0$ and $x=L$ boundary terms *cancel*. The eigenfunctions are now sines *and* cosines of $2\pi n x / L$, and give a general Fourier series (not just a sine or cosine series)! Also, \hat{A} is now negative *semidefinite* because constant u are allowed. Hence, for example, diffusion on a periodic domain still conserves total mass, because any mass that exits one side comes back in through the other side.

Further reading: The $u = v + g$ trick is closely related to the standard proof of the uniqueness of solutions to Laplace's/Poisson's equation with general Dirichlet boundaries (google "Laplace uniqueness" or "Poisson uniqueness", e.g. [this page](#)). The trick of moving boundary conditions over to the right-hand side is so obvious for finite-difference methods that it hardly has a name, but it is often commented on explicitly for finite-element methods where things are less obvious (e.g. section 3.6 of the book). There is a [review of immersed boundary methods](#) by Mittal and Iaccarino that is fairly readable, but oriented mainly towards fluid mechanics. Periodic domains arise in many cases, the most obvious being equations on a torus (e.g. waves on a membrane that loops back to itself, diffusion in a circular tube, or masses and springs connected into a ring). They also arise for systems that repeat periodically, e.g. a periodic crystal in solid-state physics, in which case you can write the solutions as [Bloch waves](#) of the form $u(\mathbf{x}) = u_{\mathbf{k}}(\mathbf{x}) e^{i\mathbf{k} \cdot \mathbf{x}}$ where $u_{\mathbf{k}}$ is a periodic function that solves a PDE with periodic boundary conditions (and plotting the eigenvalues as a function of \mathbf{k} gives a [band structure](#)).

MIT OpenCourseWare
<http://ocw.mit.edu>

18.303 Linear Partial Differential Equations: Analysis and Numerics
Fall 2014

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.