Lecture 17

Derived Green's function of ∇^2 in 3d for infinite space (requiring solutions to → zero at infinity to get a unique solution), in three steps:

1. Because the ∇^2 operator is invariant under translations (changes of variables $\mathbf{x}\to\mathbf{x}+\mathbf{y}$), showed that $G(\mathbf{x},\mathbf{x}')$ can be written as $G(\mathbf{x},\mathbf{x}')=G(\mathbf{x}-\mathbf{x}',0)$. Similarly, rotational invariance implies that $G(\mathbf{x}-\mathbf{x}',0)=g(|\mathbf{x}-\mathbf{x}'|)$ for some function $g(r)$ that only depends on the distance from $\mathbf{x}'$.
2. In spherical coordinates, solved $-\nabla^2 g = 0$ for $r > 0$ (away from the delta function), obtaining $g(r)=c/r$ for some constant c to be determined.
3. Took the distributional derivative $(-\nabla^2 g)\{\varphi\}=g\{-\nabla^2\varphi\}$ ("integrating by parts" using the fact from Lecture 7 that ∇^2 is self-adjoint) for an arbitrary test function $\varphi(\mathbf{x})$, and showed by explicit integration that we get $c\varphi(0)$. Therefore $c=1/4\pi$ for us to solve $-\nabla^2 g = \delta(\mathbf{x}-\mathbf{x}')$.

Hence $G(\mathbf{x},\mathbf{x}') = 1/4\pi|\mathbf{x}-\mathbf{x}'|$ for this problem, and $-\nabla^2 u=f$ is solved by $u(\mathbf{x})=\int f(\mathbf{x}')d^3\mathbf{x}'/4\pi|\mathbf{x}-\mathbf{x}'|$.

A physical example of this can be found in electrostatics, from 8.02: the potential V of a charge density ρ, satisfies $-\nabla^2 V=\rho/\varepsilon_0$. A point charge q at $\mathbf{x}'$ is a charge density that is zero everywhere except for $\mathbf{x}'$, and has integral q, hence is $\rho(\mathbf{x})=q\delta(\mathbf{x}-\mathbf{x}')$. Solving for V is exactly our Green's function equation except that we multiply by q/ε_0, and hence the solution is $V(\mathbf{x}) = q/4\pi\varepsilon_0|\mathbf{x}-\mathbf{x}'|$, which should be familiar from 8.02. Hence $-\nabla^2 V=\rho/\varepsilon_0$ is solved by $V(\mathbf{x})=\int\rho(\mathbf{x}')d^3\mathbf{x}'/4\pi\varepsilon_0|\mathbf{x}-\mathbf{x}'|$, referred to in 8.02 as a "superposition" principle (writing any charge distribution as the sum of a bunch of point charges).

Perhaps the most important reason to solve for $G(\mathbf{x},\mathbf{x}')$ in empty space is that solutions for more complicated systems, with boundaries, are "built out of" this one.

An illustrative example is Ω given by the 3d half-space $z>0$, with Dirichlet boundaries (solutions=0 at $z=0$). For a point $\mathbf{x}'$ in Ω, showed that the Green's function $G(\mathbf{x},\mathbf{x}')$ of $-\nabla^2$ is $G(\mathbf{x},\mathbf{x}')=(1/|\mathbf{x}-\mathbf{x}'| - 1/|\mathbf{x}-\mathbf{x}''|)/4\pi$, where $\mathbf{x}''$ is the same as $\mathbf{x}'$ but with the sign of the z component flipped. That is, the solution in the upper half-space $z>0$ looks like the solution from *two* point sources $\delta(\mathbf{x}-\mathbf{x}')-\delta(\mathbf{x}-\mathbf{x}'')$, where the second source is a "negative image" source in $z<0$. This is called the **method of images**.

Reviewed method-of-images solution for half-space. There are a couple of other special geometries where a method-of-images gives a simple analytical solution, but it is not a very general method (complicated generalizations for 2d problems notwithstanding). The reason we are covering it, instead, is that it gives an analytically solvable example of a principle that *is* general: Green's functions (and other solutions) in complicated domains *look like solutions in the unbounded domain plus extra sources on the boundaries*.

Further reading: See e.g. sections 9.5.6–9.5.8 of *Elementary Applied Partial Differential Equations* by Haberman for a traditional textbook treatment of Green's functions of ∇^2 in empty space and the half-space. If you Google "method of images" you will find lots of links, mostly from the electrostatics viewpoint see also e.g. *Introduction to Electrodynamics* by Griffiths for a standard textbook treatment; the only mathematical difference introduced by (vacuum) electrostatics is the multiplication by the physical constant ε_0 (and the identification of $-\nabla V$ as the electric field).

18.303 Linear Partial Differential Equations: Analysis and Numerics
Fall 2014