Lecture 20

3 Example: Inhomogeneity in a small volume

Suppose we are solving $-\nabla\cdot(c\nabla u) = f$ in $\Omega = \mathbb{R}^3$ with a point source $f(\mathbf{x}) = \delta(\mathbf{x}-\mathbf{x}_0)$ at $\mathbf{x}_0$. Furthermore, suppose that $c(\mathbf{x})$ is piecewise-constant as in figure 1, with $c(\mathbf{x}) = c_2$ everywhere except in a volume V, centered at $\mathbf{x}_1$, where $\mathbf{c}(\mathbf{x}) = c_1$. Now, suppose that we want the solution $u(\mathbf{x})$, but are far from V: both the source point $\mathbf{x}_0$ and the desired point $\mathbf{x}$ are far from V, with $|\mathbf{x}_1 - \mathbf{x}_0|$ and $|\mathbf{x}_1 - \mathbf{x}|$ both much bigger than the diameter of V. This is shown schematically in figure 2. In this case, we should expect the effect of the "scattered"

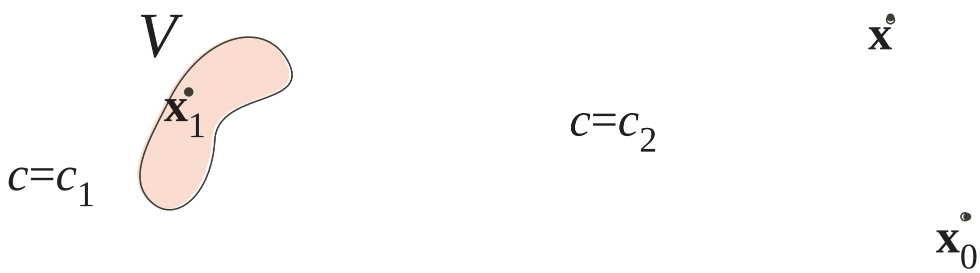

Figure 2: Schematic of problem with an inhomogeneity in a small volume V (centered at $\mathbf{x}_1$): we have a source at $\mathbf{x}_0$ and want the solution at $\mathbf{x}$, with both $\mathbf{x}_0$ and $\mathbf{x}$ much farther from $\mathbf{x}_1$ than the diameter of V.

solution from V to be small at $\mathbf{x}$, and a Born approximation should apply. Furthermore, we will assume $c_1 \approx c_2$ (though not exactly equal!), so that we can neglect the effect of the discontinuity in ∇u mentioned after equation (3) above (which greatly complicates the application of any Born-like approximation in this problem because it would prevent us from using $u \approx u_0$ in V).[3]

In this case,

$$u_0(\mathbf{x}) = G_0(\mathbf{x},\mathbf{x}_0)/c(\mathbf{x}_0) = \frac{1}{4\pi c_2|\mathbf{x}-\mathbf{x}_0|},$$

so in the Born approximation we write:

$$u(\mathbf{x}) \approx u_0(\mathbf{x}) + \hat{B}u_0,$$

where the scattered part of the solution, applying the SIE form (4) [valid when $c_1 \approx c_2$], is

$$\begin{aligned}
\hat{B}u_0 &= \ln(c_2/c_1) \oiint_{dV} G_0(\mathbf{x},\mathbf{x}')\nabla' u_0(\mathbf{x}')\cdot d\mathbf{A}' \\
&= \ln(c_2/c_1) \iiint_V \nabla'\cdot\left[G_0(\mathbf{x},\mathbf{x}')\nabla' u_0(\mathbf{x}')\right] d^3\mathbf{x}' \\
&= \ln(c_2/c_1) \iiint_V \left[\nabla' G_0(\mathbf{x},\mathbf{x}')\cdot\nabla' u_0(\mathbf{x}') + G_0 \cancel{\nabla'^2 u_0}\right] d^3\mathbf{x}',
\end{aligned}$$

[3] It turns out that many people get this wrong in electromagnetism for cases when c_1 and c_2 are very different, as discussed in my paper on a closely related subject, "Roughness losses and volume-current methods in photonic-crystal waveguides," *Appl. Phys. B* **81**, 238–293 (2005): http://math.mit.edu/~stevenj/papers/JohnsonPo05.pdf

where in the second line we applied the divergence theorem, and in the third line the product rule led to a $\nabla^2 u_0$ term, where $\nabla^2 u_0 = -\delta(\mathbf{x}-\mathbf{x}_0)$ is zero in V (since $\mathbf{x}_0$ is outside of V).

Now, since V is small compared to the distance from $\mathbf{x}$ and $\mathbf{x}_0$, the distances $|\mathbf{x}'-\mathbf{x}|$ and $|\mathbf{x}'-\mathbf{x}_0|$ hardly change for any $\mathbf{x}' \in V$, and so the $\nabla' G_0$ and $\nabla' u_0$ terms are approximately constant in this integral and we can just pull them out, giving the approximation:

$$\hat{B}u_0 \approx \ln(c_2/c_1)\ \nabla' G_0(\mathbf{x},\mathbf{x}') \cdot \nabla' u_0(\mathbf{x}')|_{\mathbf{x}'=\mathbf{x}_1}\ \text{volume}(V).$$

We can compute these gradients explicitly:

$$\nabla' \frac{1}{|\mathbf{x}'-\mathbf{y}|} = -\frac{\mathbf{x}'-\mathbf{y}}{|\mathbf{x}'-\mathbf{y}|^3},$$

and hence:

$$u(\mathbf{x}) \approx \frac{1}{4\pi c_2 |\mathbf{x}-\mathbf{x}_0|} + \ln(c_2/c_1) \frac{(\mathbf{x}_1-\mathbf{x})}{4\pi|\mathbf{x}_1-\mathbf{x}|^3} \cdot \frac{(\mathbf{x}_1-\mathbf{x}_0)}{4\pi c_2|\mathbf{x}_1-\mathbf{x}_0|^3}\ \text{volume}(V). \tag{5}$$

Notice that the amplitude of the scattered term vanishes as volume$(V) \to 0$, as expected. Notice that it also depends on the sign of $(\mathbf{x}_1-\mathbf{x}) \cdot (\mathbf{x}_1-\mathbf{x}_0)$. Why is that? What does a $\nabla' G_0$ source "mean," physically?

3.1 Dipole sources

Consider the following problem in $\Omega = \mathbb{R}^3$, requiring as usual that solutions vanish at ∞:

$$-\nabla^2 D_{\mathbf{p}}(\mathbf{x},\mathbf{x}') = -\mathbf{p}\cdot\nabla\delta(\mathbf{x}-\mathbf{x}') = +\mathbf{p}\cdot\nabla'\delta(\mathbf{x}-\mathbf{x}').$$

This is like the Green's function equation, except now we have put the *derivative* of a delta function on the right-hand side, with some constant vector $\mathbf{p}$ (the "dipole moment"). Recall what the derivative of a delta function is:

$$[-\mathbf{p}\cdot\nabla\delta(\mathbf{x}-\mathbf{x}')]\{\phi\} = [\delta(\mathbf{x}-\mathbf{x}')]\{\mathbf{p}\cdot\nabla\phi\} = \mathbf{p}\cdot\nabla\phi|_{\mathbf{x}'} = \lim_{\epsilon\to 0}\frac{\phi(\mathbf{x}'+\epsilon\mathbf{p})-\phi(\mathbf{x}'-\epsilon\mathbf{p})}{2\epsilon},$$

and hence (similar to pset 5 of 2010 or pset 7 of 2011),

$$-\mathbf{p}\cdot\nabla\delta(\mathbf{x}-\mathbf{x}') = \lim_{\epsilon\to 0}\frac{\delta(\mathbf{x}-\mathbf{x}'-\epsilon\mathbf{p})-\delta(\mathbf{x}-\mathbf{x}'+\epsilon\mathbf{p})}{2\epsilon}.$$

That is, the derivative of a delta function is a limit of limit of *two* delta functions of *opposite* sign, displaced proportional to $\mathbf{p}$. In 8.02, where delta functions are "point charges," this is what you would have called an "electric dipole."

We can solve for $\mathbf{D}_{\mathbf{p}}$ quite easily, because we know the solution G_0 to $-\nabla^2 G_0(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')$, and ∇ and ∇' derivatives can be interchanged in their order:

$$-\mathbf{p}\cdot\nabla\delta(\mathbf{x}-\mathbf{x}') = \mathbf{p}\cdot\nabla'\left[\delta(\mathbf{x}-\mathbf{x}')\right] = \mathbf{p}\cdot\nabla'\left[-\nabla^2 G_0(\mathbf{x},\mathbf{x}')\right] = -\nabla^2\left[\mathbf{p}\cdot\nabla' G_0(\mathbf{x},\mathbf{x}')\right],$$

and hence

$$D_{\mathbf{p}}(\mathbf{x},\mathbf{x}') = \mathbf{p}\cdot\nabla' G_0(\mathbf{x},\mathbf{x}') = \mathbf{p}\cdot\frac{\mathbf{x}-\mathbf{x}'}{4\pi|\mathbf{x}-\mathbf{x}'|^3}.$$

In electrostatics, ths would be the potential of a dipole. Note that this falls off as $\sim 1/|\mathbf{x}-\mathbf{x}'|^2$, whereas G_0 falls off as $\sim 1/|\mathbf{x}-\mathbf{x}'|$.

Given this solution, we can now interpret the scattered part of the solution (5) above: **a small inhomogeneity gives an effective dipole source $\mathbf{p}$** at $\mathbf{x}_1$, where

$$\mathbf{p} = -\ln(c_2/c_1)\frac{(\mathbf{x}_1-\mathbf{x}_0)}{4\pi|\mathbf{x}_1-\mathbf{x}_0|^3}\,\text{volume}(V).$$

In electrostatics, for a typical case where V is a small piece of matter in vacuum, $c_2 < c_1$, so $\mathbf{p}$ is parallel to $\mathbf{x}_1 - \mathbf{x}_0$. Physically, a positive point charge induces a dipole moment $\mathbf{p}$ *pointed away* from the charge, because a "+" charge at $\mathbf{x}_0$ pushes "+" charges in V *away* from it, as shown below.

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