Lecture 21

New topic: Time-stepping and stability. Before, we turned operator equations $\hat{A}u=f$ into matrix equations $A\mathbf{u}=\mathbf{f}$ by discretizing in space. Now, we want to turn time-dependent operator equations $\hat{A}u=f+\partial u/\partial t$ into discrete equations in both time and space. This will involve a new concern: **stability**.

Began with a trivial example of an operator $\hat{A}=a$, a single number $a<0$, which for $f=0$ gives the ODE $du/dt=au$, and has the exponentially decaying (for $a<0$) solution $u(t)=u(0)e^{at}$. Now we will discretize $u(t)$ in time by $u(n\Delta t)\approx u^n$ — we will always use *superscripts* to denote the *timestep* n. Approximating the time derivative by a *forward difference* yields $u^{n+1}\approx(1+a\Delta t)u^n=(1+a\Delta t)^{n+1}u^0$. Even though the exact ODE has decaying solutions, this discretization may have *exponentially growing* solutions unless $\Delta t<2/|a|$: the discretization is only **conditionally stable**. In contrast, a *backward difference* yields $u^{n+1}\approx(1-a\Delta t)^{-1}u^n=(1+a\Delta t)^{-1-n}u^0$, which is always exponentially decaying for $a<0$: the scheme is **unconditionally stable**.

For a more general operator $\hat{A}$, we proceed conceptually in two steps. First, we discretize in space only to yield a system of ODEs $A\mathbf{u}=\partial\mathbf{u}/\partial t$ for a matrix A (e.g. by finite differences in space). Then we discretize in time and ask what happens to the eigenvectors of A. Focused on the case where A (and $\hat{A}$) are self-adjoint and negative-definite (negative eigenvalues $\lambda<0$), as for the heat equation ($\hat{A}=\nabla^2$) with Dirichlet boundaries. In this case, showed that forward differences give an **explicit timestep** $u^{n+1}\approx(1+A\Delta t)u^n$ and are conditionally stable: we must have $\Delta t<2/|\lambda|$. In contrast, backward differences give an **implicit timestep** $u^{n+1}\approx(1-A\Delta t)^{-1}u^n$ where we must solve a linear system at each step, but are unconditionally stable (decaying for any Δt).

Some definitions:

- $\hat{A}u=\partial u/\partial t$ is **well posed** if the solution $u(\mathbf{x},t)$ is finite for any finite t and for any initial condition $u(\mathbf{x},0)$. (Note that PDEs with diverging solutions can still be well-posed, as long as they are finite at finite times, even if they are exponentially large.)
- A discretization is **consistent** if the discretization goes to $\hat{A}u=\partial u/\partial t$ as Δx and $\Delta t \to 0$.
    - If the difference between the discrete equations and $\hat{A}u=\partial u/\partial t$, the **local truncation error**, goes to zero as Δx^a and as Δt^b, then we say the scheme is "a-th order in space and b-th order in time."
- A discretization is **stable** if $\mathbf{u}^{t/\Delta t}\approx u(\mathbf{x},t)$ does *not* blow up as $\Delta t \to 0$ and $\Delta x \to 0$. (Informally, we often say it is "stable" if the solution does not blow up for any Δt, but a more precise definition has to take into account that the original PDE may have solutions that blow up as $t\to\infty$.)
    - it is **conditionally stable** if it is stable only when Δt has a certain relationship to the spatial discretization A, and in particular this usually means that Δt is constrained by some relationship with Δx.
    - it is **unconditionally stable** if it is stable for all Δt independent of Δx or A (or at least as long as A has some property like negative-definiteness).
- A discretization is **convergent** if $\mathbf{u}^{t/\Delta t}\to u(\mathbf{x},t)$ as Δx, $\Delta t \to 0$.

A very important result (stated here without proof) is the **Lax equivalence theorem**: for any consistent discretization of a well-posed linear initial-value problem, **stability implies convergence and vice versa**. If it is unstable, then it is obvious that it cannot converge: the discretization blows up but the real solution doesn't. Less obvious is the fact that *if it does not blow up, it must converge*.

The Lax theorem is very reassuring, because it turns out that it is quite difficult to prove stability in general (we usually prove necessary but not sufficient conditions in conditionally stable schemes), but if you run it and it doesn't blow up, you know it must be converging to the correct result.

The tricky case to analyze is that of conditionally stable schemes. We need to relate the eigenvalues of A to Δx in some way to obtain a useful condition on Δt.

For explicit timestepping of the heat/diffusion equation with forward differences, Δt is proportional to Δx^2, so even though the discretization is second-order in space (errors ~ Δx^2) and first-order in time (errors ~ Δt), the time and space discretization errors are comparable (or at least proportional).

On the other hand, for implicit timestepping with backward differences, Δt is independent of Δx, so the first-order accuracy in time can really limit us. Instead, presented a second-order scheme in time by considering $(\mathbf{u}^{n+1}-\mathbf{u}^n)/\Delta t$ to be a *center* difference around step n+0.5 [$t=(n+0.5)\Delta t$]. In this case, we evaluate the right-hand side $A\mathbf{u}$ at n+0.5 by averaging: $A(\mathbf{u}^{n+1}+\mathbf{u}^n)/2$. This gives a **Crank-Nicolson** scheme: $\mathbf{u}^{n+1}=(1-A\Delta t/2)^{-1}(1+A\Delta t/2)\mathbf{u}^n$. This is an implicit scheme, but is second-order accurate in both space and time (assuming a 2nd-order A in space). Showed that it is unconditionally stable if A is negative-definite.

For conditionally stable schemes, we need the eigenvalues of A. Gave a crude argument that the biggest $|\lambda|$ for ∇^2 and similar operators is proportional to Δx^2, based on the fact that the solution cannot oscillate faster than the grid. To do better than this, we need to consider simplified cases that we can analyze analytically.

18.303 Linear Partial Differential Equations: Analysis and Numerics
Fall 2014