

Lecture 22

Von Neumann analysis. The idea of Von Neumann analysis is to analyze the eigenvalues of the space discretization, A , in a simple case that can be solved analytically: ∞ space and constant coefficients. In this case the eigensolutions will be sinusoids (Fourier modes), which are most conveniently written as complex exponentials.

In particular, considered $\hat{A} = d^2/dx^2$ in one dimension, discretized by the usual center difference. We try a solution $u_m = e^{ikm}$, and show that it is indeed an eigenvector (with infinitely many components in ∞ space!) of the discretized second derivative. (Briefly reviewed properties of complex exponentials and the equivalence to sines and cosines by [Euler's identity](#).) Showed that the corresponding eigenvalues are $\lambda(k) = -4 \sin^2(k/2) / \Delta x^2$. Hence, the maximum $|\lambda|$ is for $k = \pi$ (a solution that oscillates with every grid point).

Applying the conditional stability result from last lecture for forward-difference timestepping, we find $\Delta t < \Delta x^2/2$. This is necessary and sufficient for stability in the ∞ -space case, because any (polynomially bounded) initial condition can be written as a sum of these e^{ikm} functions. In fact, this is a kind of reverse Fourier series, a [discrete-"time" Fourier transform](#) (DTFT, although here it is space not time). Reviewed how Fourier series can be written in terms of complex exponentials rather than sines and cosines. Noted that k and $k+2\pi$ give equivalent solutions: this is called [aliasing](#), and means that we only need to consider k in $[-\pi, \pi]$.

When we have boundaries, inhomogeneities, etcetera, then it is usually too hard to compute the eigenvalues exactly; in this case Von Neumann analysis usually gives us at best a necessary condition for stability, but not a sufficient condition. In practice, though, it works very well, although usually we err on the conservative side and make Δt a little bit smaller than the Von Neumann bound might strictly request.

Similarly, analyzed the 2d heat equation with center differences in space and forward differences in time, and showed that the maximum Δt is decreased by a factor of 2. In general, it is decreased proportional to the number of dimensions.

The important consequence of this is: when you refine the discretization in space (decreasing Δx), you must *also refine the discretization in time* (decreasing Δt) in an explicit scheme (like forward differences in time).

New subject: **Wave equations.** Although we originally wrote the wave equation as a second derivative in time, in order to think about time evolution (either numerically or analytically) it is nicer to write it as a first-order derivative into time. The most obvious way to do this is to introduce a new variable $v = \partial u / \partial t$, but this turns out to lead to a somewhat unsymmetrical system of equations that is hard to analyze.

Instead, we will look at the scalar wave equation $\nabla^2 u = \partial^2 u / \partial t^2$ in a new way. We will introduce a new vector-valued unknown \mathbf{v} , defined by $\partial \mathbf{v} / \partial t = \nabla u$ and $\partial u / \partial t = \nabla \cdot \mathbf{v}$; showed that this is equivalent to $\nabla^2 u = \partial^2 u / \partial t^2$. This leads to a new equation of the form $\partial \mathbf{w} / \partial t = \hat{A} \mathbf{w}$, where $\mathbf{w} = (u; \mathbf{v})$ and \hat{A} is the 2×2 operator $\hat{A} = (0, \nabla \cdot; \nabla, 0) = (0, \text{div}; \text{grad}; 0)$. Now the problem looks superficially like

the heat/diffusion equation: it is first-order in time, with some new operator \hat{A} . But this \hat{A} is very different from the old $\hat{A}=\nabla^2$! In fact, we will see that this \hat{A} gives $\hat{A}^*=-\hat{A}$: it is **anti-Hermitian**, and from this stems many of the important properties of wave equations.

Further reading: See for example chapter 6 of the Strang book for a typical textbook treatment of Von Neumann analysis. Something I dislike about this and many textbooks is that it does Von Neumann analysis right away. I prefer considering the dependence of Δt on the eigenvalues of A in general first (where things are both simpler and more general than diving into a specific discretization), and only then finding the eigenvalues of A in the Von Neumann case where we can solve for them exactly.

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