

## Lecture 23

Defined the inner product  $\langle \mathbf{w}, \mathbf{w}' \rangle$  in the obvious way as  $\int (u u' + \mathbf{v} \cdot \mathbf{v}')$  and showed that  $\hat{A}^* = -\hat{A}$ , i.e. that it is anti-Hermitian, for either Dirichlet or Neumann boundary conditions. From this, reprised the proof of real eigenvalues for the Hermitian case to show that now the eigenvalues are purely imaginary. Alternatively, showed that  $i\hat{A}$  is a Hermitian operator, so all of the properties of Hermitian operators carry over to  $\hat{A}$  except that the eigenvalues are multiplied by  $i$ .

Showed that an anti-Hermitian operator  $\hat{A}$  is simply  $i$  times a self-adjoint/Hermitian operator  $-i\hat{A}$ , and therefore it inherits the nice properties of self-adjoint operators with one difference: the eigenvalues are *imaginary* instead of real. If we call the eigenvalues  $-i\omega$  for a real  $\omega$ , then it is clear that we obtain oscillating solutions with time dependence  $e^{-i\omega t}$ . Furthermore, showed that  $\langle \mathbf{w}, \mathbf{w} \rangle$  is conserved in time, which in the next lecture we will interpret as conservation of "energy".

With the wave equation in a new form  $\partial \mathbf{w} / \partial t = D \mathbf{w}$ , we had derived important properties of  $D$ : anti-Hermitian, imaginary eigenvalues, unitary time evolution, conservation of energy, similar to the handout (but at a somewhat simpler level, as the handout is from a graduate class).

As in the notes from the previous lecture, considered the general case of the scalar wave equation with non-constant coefficients  $a, b > 0$ :  $b \nabla \cdot (a \nabla u) = \partial^2 u / \partial t^2$ , splitting this up as  $\partial \mathbf{v} / \partial t = a \nabla u$  and  $\partial u / \partial t = b \nabla \cdot \mathbf{v}$ . As in the notes, showed that the resulting  $D$  operator is still anti-Hermitian but under a modified inner product  $\langle \mathbf{w}, \mathbf{w}' \rangle = \int (u u' / b + \mathbf{v} \cdot \mathbf{v}' / a)$ .

Gave simple example of compression waves in a 1d system that is the limit of springs and masses, from lecture 5.5. If  $h$  is the displacement, it is convenient (to obtain conservation of energy in a familiar form) to write  $u = \partial h / \partial t$  and  $\mathbf{v} = \partial h / \partial x$ . We then get  $b = 1/\rho$ ,  $a = \kappa$  (a "spring constant"), and find that  $\int (\rho u^2 + \kappa v^2)$  is conserved. We interpreted this as kinetic+potential energy.

Example: pressure waves in a fluid or gas. In this case,  $u = P$  (pressure),  $\mathbf{v}$  is a velocity,  $a = 1/\rho$  ( $\rho$  is density), and  $b = K$  (bulk modulus:  $dP = -K dV/V$ , relating change in pressure  $dP$  to fractional change in volume  $dV/V$ ). Again this gives a wave equation, with a conserved kinetic+potential energy  $\int (\rho |\mathbf{v}|^2 + P^2/K)$ .

Considered the case of Maxwell's equation in vacuum (which you already proved is anti-Hermitian in homework) and gave the corresponding energy density in the EM fields. Mentioned the case of Schrodinger's equation in quantum mechanics, where we only have one time derivative, and the conserved norm is interpreted as conservation of probability. The other cases in the notes are more complicated.

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