Lecture 24

Discretization of the (1d scalar) wave equation: staggered grids and leap-frog schemes. Von Neumann and CFL analysis. Dispersion relation.

Discretization of the (1d scalar) wave equation, simplifying for now to an infinite domain (no boundaries) and constant coefficients (c=1). This corresponds to the equations $\partial u/\partial t=\partial v/\partial x$ and $\partial v/\partial t=\partial u/\partial x$.

The obvious strategy is to make everything a center difference. First concentrating on the spatial discretization, showed that this means that u and v should be discretized on different grids: for integers m, we should discretize $u(m\Delta x)\approx u_m$ and $v([m+0.5]\Delta x)\approx v_{m+0.5}$. That is, the u and v spatial grids are offset, or **staggered**, by $\Delta x/2$.

For discretizing in time, one strategy is to discretize u and v at the same timesteps $n\Delta t$. Center-differencing then leads to a Crank-Nicolson scheme, which can easily show to be unconditionally stable (albeit implicit) for anti-Hermitian spatial discretizations.

Alternatively, we can ues an explicit **leap-frog** scheme in which u is discretized at times $n\Delta t$ and v is discretized at times $[n-0.5]\Delta t$. Sketched out the corresponding staggered grids, difference equations, and leap-frog process.

Went through Von Neumann stability analysis of this leap-frog scheme, and derived the **dispersion relation** $\omega(k)$ for **planewave** solutions $e^{ik\Delta x\, m - i\omega\Delta t\, n}$. Compared to dispersion relation $\omega(k)=\pm c|k|$ of the analytical equation: matches for small k, but a large mismatch as k approaches $\pi/\Delta x$.

Further reading: Strang book, section 6.4 on the leapfrog scheme for the wave equation.

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18.303 Linear Partial Differential Equations: Analysis and Numerics
Fall 2014

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