Lecture 28

Began discussing general topic of waveguides. Defined waveguides: a wave-equation system that is *invariant* (or periodic) in at least one direction (say y), and has some structure to *confine* waves in one or more of the other "transverse" directions. A simple example of a waveguide (although not the only example) consists of waves confined in a hollow pipe (either sound waves or electromagnetic waves, where the latter are confined in metal pipe). Began with a simple 2d example: a waveguide for a scalar wave equation that is invariant in y and confines waves with "hard walls" (Dirichlet boundaries at x=0 and x=L) in the x direction. In such a wave equation, or *any wave equation that is invariant in y*, the solutions are separable in the invariant direction, and the eigenfunctions $u(x,y)e^{-i\omega t}$ can be written in the form $u_k(x)e^{i(ky-\omega t)}$ for some function u_k and some eigenvalues $\omega(k)$. In this case, plugged the separable form into the scalar wave equation and immediately obtained a 1d equation for u_k: $u_k''-k^2u_k=-\omega^2u_k$, which we solved to find $u_k=\sin(n\pi x/L)$ for $\omega^2=k^2+(n\pi/L)^2$. Plotted the dispersion relation $\omega(k)$ for a few *guided modes* (different integers n), and discussed what the corresponding modes look like.

Commented on the k goes to 0 and infinity limits where the group velocity goes to 0 and 1 (c), respectively. As k goes to zero, the group velocity goes to zero but the phase velocity diverges; discuss what this means.

Discussed superposition of modes: explain that if we superimpose say the n=1 and n=2 modes at the same ω and nearby k, what we get is a "zig-zagging" asymmetrical solution that bounces back and forth between the walls at intervals $\pi/\Delta k$. This is what we might get if we add an off-center source term, for example.

Discussed the existence of a low-ω *cutoff* for each mode and its implications. As we increase the frequency of a source term, it excites more and more modes (a quantum analogue of this phenomenon is quantized conductance in nanowires!). Moreover, by Taylor-expanding the dispersion relation near the cutoff as a quadratic function, we can solve for the solutions slightly *below* cutoff, and see that they must have *imaginary k* and hence be *exponentially decaying/growing*. These are called **evanescent modes** (as opposed to propagating modes for real k), and can only be excited by a localized source or some break or boundary in the waveguide (e.g. an endfacet); they are what you get if you try to vibrate a membrane below cutoff!

Waveguide movies: for a 2d waveguide of width L, put an off-center source at one end that turns on around t=0 to a sinusoidal forcing of frequency $f=\omega\cdot L/2\pi c$, and showed some movies of computer simulations. First, considered a waveguide with hard ("metal") walls like the previous example; depending on how f relates to the mode cutoffs (at 0.5, 1.0, 1.5, ...), we get very different results. Then, considered a source in an infinite homogeneous (c=1) medium ("vacuum"), which just gives waves radiating outwards in every direction. Finally, considered a medium that is c=1 in a width L, and outside is c=2: this gives waveguiding by a very different mechanism, "total internal reflection".

18.303 Linear Partial Differential Equations: Analysis and Numerics
Fall 2014