Lecture 29

Went through the 2d example from the previous lecture analytically; see handout for notes. In general, we need both propagating and evanescent waves in order to find the solution when we have something that breaks translational symmetry (such as a localized source).

Waveguide modes, in general: In a waveguide, or any system that is invariant along one dimension (say z), we can always find *separable eigenfunctions* of $\hat{A}u=\partial^2 u/\partial t^2$. That is, we look for solutions of the form $u_k(x,y)\, e^{i(kz-\omega t)}$, which are eigenfunctions of $\hat{A}$ with eigenvalue $-\omega^2$. These are solutions to the full problem, with *each value of k giving us different solutions* u_k and $\omega(k)$. We can then build any arbitrary solution u via a superposition of these (much like a Fourier transform, writing any z dependence as a sum of e^{ikz} sinusoids). For each k, the function $u_k(x,y)$ (which does not depend on y) satisfies $\hat{A}e^{ikz}u_k=-\omega(k)^2 e^{ikz}u_k$, from which we derive the **reduced eigenproblem** $\hat{A}_k u_k=-\omega(k)^2 u_k$, where $\hat{A}_k=e^{-ikz}\hat{A}e^{ikz}$ is an operator with no z derivatives and no z dependence: we have reduced the problem to one fewer spatial dimension.

Showed that $\hat{A}_k$ is self-adjoint and definite if $\hat{A}$ is.

A **waveguide** is any system in at which at least *some* of these modes u_k are *localized* (or *guided*): in particular, we usually require them to be at least square-integrable (finite-norm) over the reduced (x,y) domain (this is always true if the reduced domain is finite as for our "tube" waveguide examples). (In practice, guided modes usually decay at least exponentially fast in |(x,y)| outside of some compact region.)

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