Lecture 30

Guidance, reflection, and refraction at interfaces between regions with different wave speeds c:

Started with the solutions of the scalar wave equation in infinite space with a constant coefficient (speed) c: plane waves $u(\mathbf{x},t)=e^{i(\mathbf{k}.\mathbf{x}-\omega t)}$, satisfying $\omega=c|\mathbf{k}|$, where $\mathbf{k}$ is the *wavevector* and indicates the propagation direction and the spatial wavelength $2\pi/|\mathbf{k}|$.

Now, considere what happens when a plane wave in a region with speed c_1 is incident upon an interface at x=0 to another region with speed c_2. In general, we expect a transmitted wave and a reflected wave. At x=0, we will have some continuity conditions depending on the specifics of the wave equation (e.g. u continuous), and these continuity conditions must be *satisfied at all y and at all t*. The only way to satisfy the same continuity conditions at all y is for all of the waves to be oscillating at the same speed in the y direction at x=0, i.e. that they must all have the same k_y, and the only way to satisfy the same continuity conditions at all t is for the waves to be oscillating at the same ω. Writing $k_y=|\mathbf{k}|\sin\theta=(\omega/c)\sin\theta$, we immediately obtain two results. First, the reflected angle is the same as the incident angle. Second, $(1/c_1)\sin\theta_1=(1/c_2)\sin\theta_2$. In optics, these are known as the **Law of Equal Angles** and **Snell's Law** respectively, but they are generic to *all* wave equations.

If $c_1<c_2$, then showed that there are no real θ_2 solutions for a sufficiently large angle θ_1. In optics, you probably learned this as **total internal reflection**, but it is general to any wave equation. Then, if we have two interfaces, with $c_1<c_2$ sandwiched between two semi-infinite c_2 regions, we can obtain *guided modes* that are trapped mostly in c_1, and can crudely be thought of as "rays" bouncing back and forth in c_1, "totally internally reflected". More carefully, showed that "totally internally reflected" solutions correspond to **exponentially decaying solutions** in c_2, which are called *evanescent waves*.

To obtain a more general picture, we imagine writing down the dispersion relation $\omega(k)$ for such a waveguide, looking as usual for separable eigenfunctions $u_k(x)e^{i(ky-\omega t)}$. Far from the c_1 region, the solutions must just be planewaves propagating in c_2, with $\omega=c_2|\mathbf{k}|=c_2k\sec\theta$, since k is just the y component of $\mathbf{k}$, where θ is the angle with the y axis. Plotting all of these solutions forms a continuous **cone** covering $\omega(k)\geq c_2k$ (called the "light cone" in optics): this cone is *all the wave solutions that propagate in c_2*. The light cone for the c_1 region has a lower slope (c_1), and hence the c_1 region will introduce new *guided* solutions below the c_2 cone which are evanescent in c_2. In the next lecture, I will argue that a finite-thickness c_1 region leads to a finite number of guided modes below the c_2 cone, and give numerical examples.

Further reading: You can find many explanations of Snell's law, total internal reflection, etcetera, online. For a treatment in the context of the scalar wave equation, see e.g. *Haberman, Elementary Applied Partial Differential Equations* section 4.6. For a treatment in Maxwell's equations, see any elementary electromagnetism book; our book (chapter 3) has an abstract approach with a light cone etcetera mirroring the one here.

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