

Lecture 34

Began to introduce a new topic, **finite element methods**.

Set up the two key components of finite element methods (FEM): the basis and the discretization of the equations. FEM generalizes finite difference methods to nonuniform meshes, meshes that can conform to boundaries/interfaces, and gives more freedom in having different equations/discretizations/bases in different regions. A typical 2d mesh is composed of triangles (possibly with curved edges), in 3d tetrahedra, and in 1d just line segments, although other polyhedra are also possible. The vertices of the triangles/tetrahedra/etc are called *nodes* of the mesh.

First, we write unknown functions $u(x)$ (in some Sobolev space V) approximately as $\tilde{u}(x)$ (in an N -dimensional space \tilde{V}) in terms of N basis functions $b_n(x)$ (spanning \tilde{V}) multiplied by unknown coefficients u_n , defined with respect to the mesh. Typically, the $b_n(x)$ are simple (low degree) polynomial functions defined piecewise in each element, with some continuity constraints.

For example, gave the example of "tent" functions (1st order elements) where $b_n(x)$ is 1 at node n , 0 at other nodes, and linearly interpolated between node n and adjacent nodes. For this basis, u_n is just the value of $\tilde{u}(x)$ at the nodes. In the 1d case this just corresponds to linearly interpolating $\tilde{u}(x)$ between each pair of nodes.

Given a basis, we need to construct a discretization of $\hat{A}u=f$, and the typical way to do this is a **Galerkin** method. Recall that, in the distributional/weak sense, the exact solution satisfies $\langle \varphi, \hat{A}u \rangle = \langle \varphi, f \rangle$ for test functions φ . (Typically, we phrase the problem in terms of the **bilinear form** $\langle \varphi, u \rangle_A = \langle \varphi, \hat{A}u \rangle$, where we usually integrate by parts so that half of the derivatives fall on φ . This avoids the need for explicit delta functions. e.g. for $\hat{A} = -\nabla^2$ with Dirichlet boundaries we get $\langle \varphi, u \rangle_A = \langle \nabla \varphi, \nabla u \rangle$.) Here, we need to get just N equations, so we will do this for only N test functions φ , and in particular the Galerkin approach is to choose $\varphi = b_m$ for m from 1 to N . Showed that this gives a matrix equation $Au=f$, where the entries of u are the unknown coefficients u_n , with:

- The entries of f are $f_n = \langle b_n, f \rangle$
- The entries of A are $A_{mn} = \langle b_m, b_n \rangle_A$.

For the integrals of A_{mn} to exist, some continuity constraints must be imposed on the basis. For example, with $\hat{A} = \nabla^2$ or similar 2nd-derivative operators, we only need b_n to be continuous and piecewise differentiable.

Another way of looking at the Galerkin approach is that we require the *residual* $\hat{A}\tilde{u}-f$ to be orthogonal to b_m (i.e. the residual is orthogonal to \tilde{V}). As we increase N , and the basis approaches a complete basis, the intuition is that this forces the error to go to zero (in the distributional/weak sense). Later, we will outline a more careful convergence proof.

For \hat{A} self-adjoint and positive-definite, the bilinear form $\langle u, v \rangle_A$ is a proper inner product and $\|u\|_A = \langle u, u \rangle_A^{1/2}$ is a norm. From $\langle b_m, \tilde{u} \rangle_A = \langle b_m, f \rangle = \langle b_m, u \rangle_A$ where u is the exact solution, we

obtain $\langle b_m, \tilde{u}-u \rangle_A = 0$. That is, the error $\tilde{u}-u$ is orthogonal to \tilde{V} in the $\langle \cdot, \cdot \rangle_A$ sense. It follows that \tilde{u} is the *orthogonal projection* of u onto \tilde{V} : \tilde{u} minimizes $\|\tilde{u}-u\|_A$ over all \tilde{u} in \tilde{V} . This is a key ingredient of convergence proofs.

Showed that Galerkin discretizations preserve some nice properties of \hat{A} : if \hat{A} is self-adjoint, then A is Hermitian; if \hat{A} is positive-definite (or negative-definite, etcetera) then A is positive-definite (or negative-definite, etcetera).

Further reading: Textbook, section 3.6. There are lots of books on finite-element methods. These [Finite Element Analysis Course Notes](#) by Joseph E. Flaherty at RPI are pretty helpful.

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