

## Lecture 4

Showed that our  $A$  indeed has this  $-D^T D$  form and hence is negative-definite: we derived the discrete Laplacian by turning two derivatives into differences, one by one, and now by writing the first step as a matrix we get  $D$ , while writing the second step as a matrix shows that it is  $-D^T$ . To get a negative *definite* matrix (as opposed to just negative semidefinite), we additionally require that  $D$  be full column rank; showed that this is easy to see from  $D^T$  since it is upper-triangular.

To do a similar analysis of the actual Laplacian, we first have to have a dot product, or **inner product**. Defined an abstract  $\langle u, v \rangle$  notation (a map from *functions*  $u$  and  $v$  to *scalars*  $\langle u, v \rangle$ ) for inner products, as well as three key properties. First,  $\langle u, v \rangle = \text{complex conjugate of } \langle v, u \rangle$ . Second,  $|u|^2 = \langle u, u \rangle$  must be nonnegative, and zero only if  $u=0$ . Third, it must be linear:  $\langle u, \alpha v + \beta w \rangle = \alpha \langle u, v \rangle + \beta \langle u, w \rangle$ . (Note: some textbooks, especially in functional analysis, put the conjugation on the second argument instead of the first.) For functions, the most common inner product (though *not the only choice* and not always the *best* choice, as we will see next time) is a simple integral  $\int uv$  (conjugating  $u$  for complex functions); we will look at this more next time.

Reviewed inner products of functions. A vector space with an inner product (plus a technical criterion called "completeness" that is almost always satisfied in practice) is called a **Hilbert space**. Note that we include only functions with finite norm  $\langle u, u \rangle$  in the Hilbert space (i.e. we consider only **square-integrable** functions), which throws out a lot of divergent functions and means that everything has a convergent Fourier series. (Another omitted technicality: we have to ignore finite discrepancies at isolated points, or otherwise you can have  $\langle u, u \rangle = 0$  for  $u(x)$  nonzero; there is a rigorous way to do this, which we will come back to later.)

Defined the **adjoint**  $\hat{A}^*$  of a linear operator: whatever we have to do to move it from one side of the inner product to the other, i.e. whatever  $\hat{A}^*$  satisfies  $\langle u, \hat{A}v \rangle = \langle \hat{A}^*u, v \rangle$  for all  $u, v$ . (Omitted technicality: we must further restrict ourselves to functions that are sufficiently differentiable that  $\langle u, \hat{A}u \rangle$  is finite, which is called a **Sobolev space** for this  $\hat{A}$ , a subset of the Hilbert space.) For matrices and ordinary vector dot products, this is equivalent to the "swap rows and columns" definition. For differential operators, it corresponds to integration by parts, and depends on the boundary conditions as well as on the operator and on the inner product.

Showed that with  $u(0)=u(L)=0$  boundary conditions and this inner product,  $(d^2/dx^2)^T$  is real-symmetric (also called "Hermitian" or "self-adjoint"). [There is an omitted technicality here: technically, we have only showed that the operator is symmetric. To show that it is Hermitian, we must also show that the adjoint has the same **domain** in the Hilbert space. Mostly we can avoid this technical distinction in real applications; it doesn't arise explicitly in the proofs here.]

Not only that, but next time we will show that  $d^2/dx^2$  is negative-definite on this space, since  $\langle u, u'' \rangle = -\int |u'|^2$ , and  $u'=0$  only if  $u=\text{constant}=0$  with these boundary conditions.

Showed that the proof of real eigenvalues from 18.06 carries over without modification for Hermitian operators; similarly for the proof of orthogonal eigenvectors, hence the orthogonality of the Fourier sine series. Similarly for the proof of negative eigenvalues.

So, many of the key properties of  $d^2/dx^2$  follow "by inspection" once you learn how to transpose operators (integrate by parts). And this immediately tells us key properties of the solutions, if we assume the spectral theorem: Poisson's equation has a unique solution, the diffusion equation has decaying solutions, and the wave equation has oscillating solutions.

**Further reading:** [Notes on function spaces, Hermitian operators, and Fourier series](#) that I once wrote for 18.06 (slightly different notation). Textbook, section 3.1: transpose of a derivative. The general topic of linear algebra for functions leads to a subject called *functional analysis*; a rigorous introduction to functional analysis can be found in, for example, the book *Basic Classes of Linear Operators* by Gohberg et al. There are some technicalities that I omit: a differential operator is only called "self-adjoint" if it is equal to its adjoint and is "densely defined", and showing that an operator equals its adjoint furthermore requires an extra step of showing that  $\hat{A}$  and  $\hat{A}^*$  act on the same domains.

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